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TOPICAL REVIEW

Algebraic Bethe ansatz method for the exact calculation of energy spectra and form factors: applications to models of Bose–Einstein condensates and metallic nanograins

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Abstract

In this review we demonstrate how the algebraic Bethe ansatz is used for the calculation of the energy spectra and form factors (operator matrix elements in the basis of Hamiltonian eigenstates) in exactly solvable quantum systems. As examples we apply the theory to several models of current interest in the study of Bose–Einstein condensates, which have been successfully created using ultracold dilute atomic gases. The first model we introduce describes Josephson tunnelling between two coupled Bose–Einstein condensates. It can be used not only for the study of tunnelling between condensates of atomic gases, but for solid state Josephson junctions and coupled Cooper pair boxes. The theory is also applicable to models of atomic–molecular Bose–Einstein condensates, with two examples given and analysed. Additionally, these same two models are relevant to studies in quantum optics. Finally, we discuss the model of Bardeen, Cooper and Schrieffer in this framework, which is appropriate for systems of ultracold fermionic atomic gases, as well as being applicable for the description of superconducting correlations in metallic grains with nanoscale dimensions. In applying all the above models to physical situations, the need for an exact analysis of small-scale systems is established due to large quantum fluctuations which render mean-field approaches inaccurate.

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Contents

1. Introduction

Exactly solvable models of quantum mechanical systems provide an important insight into the nature of quantum physics, with the simple harmonic oscillator and the non-relativistic hydrogen atom serving as the archetypal examples. One method for solving these models is to exploit an underlying algebraic structure, well known to be the Lie algebra $gl(3)$ for the harmonic oscillator and $so(4)$ for the hydrogen model [1]. In this approach, the Lie algebraic structure plays the role of generating states of the system while at the same time providing state labels (or quantum numbers). A celebrated exact solution of a quantum many-body model is that for the one-dimensional Heisenberg (spin-$1/2$) chain, due to Bethe [2]. Out of this work grew the concept of the Bethe ansatz for the construction of the eigenvectors for an exactly solvable Hamiltonian. In adopting this method, a general possible form for an eigenvector is assumed that is dependent on several free parameters. Constraints are then determined for the parameters which ensure that this vector is an eigenvector of the Hamiltonian. The constraint equations are referred to as the Bethe ansatz equations of the model.

Motivated by Bethe's work the field of exactly solvable models flourished during the 1960s led by McGuire [3], Lieb [4], Sutherland [5], Yang [6] and Baxter [7], amongst many others. Out of this activity arose the Yang–Baxter equation, the solution of which provides a sufficiency condition to construct a model which is exactly solvable (applicable to

one-dimensional quantum spin chains, including quantum field theories as the lattice spacing goes to zero, and classical two-dimensional lattice systems) [8, 9]. A fundamental feature of the Yang–Baxter equation is that it can always be used to construct a family of mutually commuting matrices, known as transfer matrices, which facilitates the application of the Bethe ansatz. The method of the Bethe ansatz can take a variety of forms, commonly known as the coordinate, analytic, functional and algebraic forms. It is this latter approach that will be the focus of our work here, as this is the most appropriate to serve our requirements.

The algebraic formulation of the Bethe ansatz, and the associated quantum inverse scattering method, was primarily developed by the group of mathematical physicists in St Petersburg [10–14]. Its applicability extends beyond the study of one-dimensional spin chains, quantum field theory and two-dimensional lattice models to systems of correlated electrons [15], conformal field theory [16], as well as precipitating the notion of quantum algebras (deformations of universal enveloping algebras of Lie algebras) [17–20]. The main motivation for the algebraic formulation of the Bethe ansatz was not only for calculating the energy spectrum of a model, but to also accommodate the calculation of correlation functions. An initial step in this direction is to compute the form factors of an operator (not necessarily observable). Here the term 'form factors of an operator' simply refers to the matrix elements of that operator in the basis of Hamiltonian eigenstates. Expectation values of observable operators and general correlation functions are expressible in terms of form factors through completeness relations.

The study of correlation functions in the context of exactly solvable models has its origins in Baxter's corner transfer matrix method [7]. Following on from this, a rich theory has been developed using ideas taken from affine quantum algebras, vertex operators, integrable field theories, the off-shell Bethe ansatz and the Knizhnik–Zamolodchikov equation [21–27], as well as the algebraic Bethe ansatz approach (e.g. [13, 28–31]). For the models we will study here the calculation of form factors will be undertaken through extensive use of the Slavnov formula [32] for the scalar products of Bethe eigenstates. The Slavnov formula provides an explicit determinant representation for the scalar products. A refined proof of this result was given by Kitanine *et al* [29], using the notion of factorizing the solutions of the Yang–Baxter equation in terms of Drinfeld twists [33]. They applied this method to successfully compute form factors for the anisotropic (XXZ) Heisenberg chain [29], and in a closely related work Korepin and Slavnov computed form factors for the quantum non-linear Schrödinger equation [30]. In both cases the results are valid for finite size systems, and thus this approach is appropriate for applications to nanoscale systems. The results presented here are largely inspired by these works.

The need to appeal to the exact solution of a model has been well illustrated in the context of the energy spectrum of metallic grains of nanoscale size. Experiments conducted by Ralph, Black and Tinkham (RBT) [34, 35] using single electron tunnelling spectroscopy on aluminium grains with mean radii in the range 5–13 nm indicated significant parity effects due to the number of electrons in the system. The electron number remains fixed due to the large charging energy of the grains, which is a consequence of their small size. For grains with an odd number of electrons, the gap in the energy spectrum reduces with increasing size of the system, in contrast to the case of a grain with an even number of electrons, where a gap larger than the single electron energy levels persists. In the latter case the gap can be closed by a strongly applied magnetic field. The conclusion drawn from these results is that pairing interactions are prominent in these nanoscale systems. For a grain with an odd number of electrons there will always be at least one unpaired electron, so it is not necessary to break a Cooper pair in order to create an excited state. For a grain with an even number of electrons, all excited states have at least one broken Cooper pair, resulting in a gap in the spectrum.

In the presence of a strongly applied magnetic field, it is energetically more favourable for a grain with an even number of electrons to have broken pairs, and hence in this case there are excitations which show no gap in the spectrum.

A naive approach to describe these nanograins is to apply the theory of superconductivity due to Bardeen, Cooper and Schrieffer (BCS) [36]. Indeed, the BCS model is appropriate for these systems but the associated mean-field treatment fails. There are two main reasons for this. First is because the BCS analysis makes use of the grand canonical ensemble whereas in the experiments the electron number is fixed. Second is because a mean-field theory approximates certain operators in the model by an average value. At the nanoscale level, the quantum fluctuations are sufficiently large enough that this approximation is invalid. In systems where the mean single particle energy level spacing, which is inversely proportional to the volume, is comparable to the bulk superconducting gap (as is for metallic nanograins), it was thought that pairing interactions would not correlate any energy levels. This was conjectured by Anderson [37] on the basis of the BCS mean-field analysis, but the experiments of RBT show this not to be the case. Consequently, an exact solution was desired in order to clarify the issue.

Remarkably, the exact solution of the reduced BCS model ('reduced' refers to the fact that only zero-momentum Cooper pairs are considered and all couplings for scattering of Cooper pairs are equal) had been obtained and analysed many years earlier in a series of works by Richardson and Sherman [38, 39] using an approach equivalent to the coordinate Bethe ansatz. The motivation for their work was for application of pairing interactions in nuclear systems, which involve fixed particle number, and thus the BCS analysis referred to above is not valid. However, the condensed matter physics community was unaware of this earlier work at the time the results of RBT were communicated. It was subsequently shown that theoretical results obtained through an analysis of the exact solution for the reduced BCS Hamiltonian were compatible with the experimental results of RBT [40].

One of the most currently active fields is the study of Bose–Einstein condensates of ultracold atomic gases [41, 42]. The Bose–Einstein condensed state is of a purely quantum mechanical nature and, in analogy with the phenomena of superconducting metallic nanograins discussed above, a mean-field analysis of small-scale systems composed of Bose–Einstein condensates is inadequate due to significant quantum fluctuations. While there are many novel physical properties to be explored in the study of Bose–Einstein condensates, there are three features that we will discuss here. The first is the phenomenon of Josephson tunnelling between two coupled Bose–Einstein condensates. Recall that the Josephson effect was first proposed in relation to the tunnelling of Cooper pairs through an insulating barrier separating two superconductors [43, 44]. (A very informative historical account is given in [45].) It has been proposed as a means to couple qubits for the purpose of quantum computation [46, 47]. The experimental realization of Bose–Einstein condensation in the atomic alkali gases provides a framework in which to observe macroscopic tunnelling in a system with tunable couplings. An extensive account of this phenomenon can be found in [48], which discusses in detail the canonical Josephson Hamiltonian (equivalent to a two-site Bose–Hubbard model) for the description of this effect. It is not well known that this model is exactly solvable through the quantum inverse scattering method, which was established about a decade ago in the context of the discrete self-trapping dimer model [49, 50]. Below, we show that a slightly more general model is also exactly solvable and we derive explicit exact form factors for the generalized model.

The second aspect of Bose–Einstein condensation we will discuss is that of a condensate composed of a coherent superposition of atomic and molecular states. This phenomenon has been predicted and studied by theorists (e.g., see [51–55]) and recently realized experimentally

[56–58]. In particular, for the experiment of [58] using ^{85}Rb atoms, which are converted into diatomic molecules via a Feshbach resonance, the system was prepared, allowed to evolve, and then a measurement made to determine the number of atoms in the system. By performing this procedure over different evolution times, it was established that the expectation value for the number of atoms displayed an oscillatory behaviour, indicating that the state of the system was a quantum mechanical superposition of atomic and molecular states, as opposed to a classical mixture. The result is significant in that the state of the system is composed of a superposition of two chemically distinct components.

Finally, we will also analyse the reduced BCS model, which is relevant not only for metallic nanograins as described above, but also for the study of ultracold fermionic atomic gases [59]. As is well known, for an ultracold fermionic gas the Pauli principle prohibits all particles occupying the lowest energy level. The lowest possible energy of the system is obtained by filling the Fermi sea. However, in analogy with metals it is believed that fermionic gases should be able to form Cooper pairs, and as a consequence, undergo a phase transition at a suitably low temperature into a fermionic condensate [60].

The aim of this exposition is to illustrate that the algebraic approach to the study of exactly solvable models is a rich and elegant theory with wide applicability. In particular we show how the theory applies to the systems of Bose–Einstein condensates and the reduced BCS model discussed above. Some of these results have already been communicated [61–66], while other results we will present are new. In each case we will determine the energy spectrum, as well as the form factors for the computation of correlation functions, in terms of the Bethe ansatz solution. Certain correlation functions can in fact be deduced directly from the energy spectrum by using the Hellmann–Feynman theorem [67, 68]. Examples of this procedure applied to the models discussed here can be found in [63–65]. Typically however, the form factor approach is required to build general formulae for expectation values and correlation functions. As a potential application of these results we point to the problem of quantifying entanglement in the theory of quantum information. The role of correlation functions in the characterization of entanglement has been discussed in [69–72].

Throughout, we have endeavoured to provide as much technical detail as possible for the benefit of non-experts. The exceptions are the Slavnov formula for the scalar product of states, the proof of which is beyond the scope of this review. For the proof we refer the interested reader to [29]. Also, the orthogonality of the Bethe eigenstates will not be proved. Details of this result can be found in [30]. The format of the review is as follows. We begin in section 2 with a description of the four models we will examine. In section 3, we recall the basic features of the quantum inverse scattering method for the construction of exactly solvable models. While there already exist several excellent surveys of this approach [10–14], we give a detailed account here in order to fix notation and conventions and make the review self-contained. The central aspect is the introduction of the Yang–Baxter algebra associated with the Lie algebra $gl(2)$, which is a quadratic algebra. Several examples of realizations are given. We show that in a particular limit, called the quasi-classical limit, the Yang–Baxter algebra reduces to a Lie algebra, called the Gaudin algebra. Through a realization of the Yang–Baxter algebra, the transfer matrix is constructed which leads to an exactly solvable model. We also discuss a natural $\mathbb{Z}$-graded structure of the Yang–Baxter algebra which will be exploited in later constructions. Section 4 deals with the algebraic Bethe ansatz method in a general context for the determination of the spectrum of the transfer matrix. Section 5 presents the Slavnov formula for the scalar products of the states which arise in the algebraic Bethe ansatz method of solution. We also discuss how, through the use of the Slavnov formula, the form factors for the elements of the Yang–Baxter algebra can be obtained. Section 6 turns to calculating the explicit exact solutions for the models. Formulae for the energy spectrum

are determined, which are parametrized in terms of the roots of the Bethe ansatz equations. Section 7 deals with the computation of form factors for each of the models introduced. In all cases it is necessary to first consider the solution to the inverse problem, which involves expressing a given operator in terms of the elements of the Yang–Baxter algebra. This needs to be studied on a case by case basis. Once this is achieved, the form factors for that operator can be determined. Concluding remarks are given in section 8.

2. Model Hamiltonians

Here we present, and give a description of, three models for Bose–Einstein condensates and the reduced BCS model. Our main objective is to establish that each model is exactly solvable through the algebraic Bethe ansatz. Throughout, there are no constraints imposed on the coupling parameters for all models other than they are real, which is to ensure Hermiticity.

2.1. A model for two Josephson coupled Bose–Einstein condensates

Consider the following general Hamiltonian describing Josephson tunnelling between two coupled Bose–Einstein condensates:

$$H = U_{11}N_1^2 + U_{12}N_1N_2 + U_{22}N_2^2 + \mu_1N_1 + \mu_2N_2 - \frac{\mathcal{E}_J}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right) \tag{1}$$

where the operators $a_i, a_i^\dagger, N_i = a_i^\dagger a_i$ are associated with two Heisenberg algebras with relations

$$\left[a_i, a_j^\dagger\right] = \delta_{ij} \qquad [a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0.$$

The Hilbert space of states is given by the infinite-dimensional Fock space spanned by the vectors

$$|m, n\rangle = \left(a_1^\dagger\right)^m \left(a_2^\dagger\right)^n |0\rangle \qquad m, n = 0, 1, 2, \ldots, \infty. \tag{2}$$

The model describes Josephson tunnelling between two condensates with tunnelling strength $\mathcal{E}_J/2$, the parameters U_{ij} are the amplitudes for S-wave scattering and μ_i are chemical potentials. The Hamiltonian commutes with the total particle number $N = N_1 + N_2$.

The above Hamiltonian under the constraint $U = U_{11} = U_{22} = -U_{12}/2$ has been studied widely using techniques other than the exact solution [48, 84–88]. For this case it is useful to divide the parameter space into three regimes, namely Rabi ($U/\mathcal{E}_J \ll N^{-1}$), Josephson ($N^{-1} \ll U/\mathcal{E}_J \ll N$) and Fock ($N \ll U/\mathcal{E}_J$). In the Rabi and Josephson regions one expects coherent superposition of the two condensates to be possible whereas in the Fock region the two condensates will be, in some sense, localized. There is a correspondence between (1) and the motion of a pendulum [48]. In the Rabi and Josephson regions this motion is semiclassical (i.e., the energy level spacings are of order less than N) in contrast to the Fock case. For both the Fock and Josephson regimes the analogy corresponds to a pendulum with fixed length, while in the Rabi regime the length varies. An important problem is to study the behaviour in the crossover regimes, particularly between the Josephson and Fock regimes which are the most likely to occur in an experimental context [48]. A reliable method of doing this is through the exact solution. The motivation to extend the solution to the case where the couplings U_{11}, U_{22}, U_{12} for the S-wave scattering terms can be chosen arbitrarily is for the description of a pair of Cooper pair boxes with capacitive coupling [46]. In the limit $U_{22} \to 0$, then $\langle N_2\rangle \gg \langle N_1\rangle$, in which case the model can be considered as a single Cooper pair box coupled to a reservoir.

2.2. A model for homo-atomic–molecular Bose–Einstein condensates

Next we turn our attention to a two-mode model for an atomic–molecular Bose–Einstein condensate with identical atoms. The Hamiltonian takes the form

$$H = U_{aa}N_a^2 + U_{ac}N_aN_c + U_{cc}N_c^2 + \mu_aN_a + \mu_cN_c + \Omega(a^\dagger a^\dagger c + c^\dagger aa) \tag{3}$$

which acts on a basis of Fock states analogous to (2). Here, $a^\dagger$ is the creation operator for an atomic mode while $c^\dagger$ creates a molecular mode. The parameters U_{ij} again describe S-wave scattering, μ_i are chemical potentials and Ω is the amplitude for interconversion of atoms and molecules. The Hamiltonian commutes with the total atom number $N = N_a + 2N_c$.

In the limit $U_{aa} = U_{ac} = U_{cc} = 0$ this model was studied in [54], and analysed numerically in [65] based on the Bethe ansatz solution. However, in order to compare with experimental results, in which the S-wave scatterings are significant, one needs to analyse (3) in its full generality. Estimates for the S-wave scattering parameters in the case of ^{87}Rb are given in [55].

2.3. A model for hetero-atomic–molecular Bose–Einstein condensates

The previous model can be extended to describe an atomic–molecular Bose–Einstein condensate with two distinct species of atoms, denoted by a and b, which can combine to produce a molecule c. For this case the Hamiltonian takes the form

$$\begin{aligned} H = {} & U_{aa}N_a^2 + U_{bb}N_b^2 + U_{cc}N_c^2 + U_{ab}N_aN_b + U_{ac}N_aN_c + U_{bc}N_bN_c + \mu_aN_a \\ & + \mu_bN_b + \mu_cN_c + \Omega(a^\dagger b^\dagger c + c^\dagger ba) \end{aligned} \tag{4}$$

which commutes with $\mathcal{I} = N_a - N_b$ and the total atom number $N = N_a + N_b + 2N_c$. Here the model acts on the Fock space spanned by the vectors

$$|l, m, n\rangle = (a^\dagger)^l(b^\dagger)^m(c^\dagger)^n|0\rangle.$$

Let us point out that in the limit $U_{aa} = U_{bb} = U_{cc} = U_{ab} = U_{ac} = U_{bc} = 0$, equation (4) is the Hamiltonian studied in [89, 90] modelling second harmonic generation in quantum optics. Non-zero values of these parameters correspond to a Kerr effect.

2.4. The reduced BCS model

The physical properties of a metallic nanograin with pairing interactions are described by the reduced BCS Hamiltonian [40]

$$H = \sum_{j=1}^{\mathcal{L}} \epsilon_j n_j - g \sum_{j,k=1}^{\mathcal{L}} c_{k+}^\dagger c_{k-}^\dagger c_{j-} c_{j+}. \tag{5}$$

Above, $j = 1, \ldots, \mathcal{L}$ labels a shell of doubly degenerate single particle energy levels with energies ϵ_j and $n_j = c_{j+}^\dagger c_{j+} + c_{j-}^\dagger c_{j-}$ is the fermion number operator for level j. The operators $c_{j\pm}, c_{j\pm}^\dagger$ are the annihilation and creation operators for the fermions at level j. The labels $\pm$ refer to time-reversed states.

One of the features of the Hamiltonian (5) is the *blocking effect*. For any unpaired fermion at level j the action of the pairing interaction is zero since only paired fermions are scattered. This means that the Hilbert space can be decoupled into a product of paired and unpaired fermion states in which the action of the Hamiltonian on the space for the unpaired fermions is automatically diagonal in the natural basis. In view of the blocking effect, it is convenient to introduce hard-core boson operators $b_j = c_{j-}c_{j+}$, $b_j^\dagger = c_{j+}^\dagger c_{j-}^\dagger$ which satisfy the relations

$$\left(b_j^\dagger\right)^2 = 0 \qquad \left[b_j, b_k^\dagger\right] = \delta_{jk}\left(1 - 2b_j^\dagger b_j\right) \qquad [b_j, b_k] = \left[b_j^\dagger, b_k^\dagger\right] = 0 \tag{6}$$

on the space excluding single particle states. In this setting the hard-core boson operators realize the $su(2)$ algebra in the pseudo-spin representation, which will be utilized below.

The original approach of Bardeen *et al* [36] to describe the phenomenon of superconductivity in a bulk system was to employ a mean-field theory using a variational wavefunction for the ground state

$$|\Psi\rangle = \prod_{i=1}^{\mathcal{L}} \left(u_i I + v_i b_i^\dagger\right)|0\rangle \tag{7}$$

which has an undetermined number of electrons. The expectation value for the number operator is then fixed by means of a chemical potential term μ, i.e. the grand canonical ensemble is used. One of the predictions of the BCS theory is that the number of Cooper pairs in the ground state of the system is given by the ratio Δ/d where Δ is the BCS 'bulk gap' and d is the mean level spacing for the single electron energies. For nanoscale systems, this ratio is of the order of unity, in seeming contradiction with the experimental results discussed above. The explanation for this is that the mean-field approach is inappropriate in this instance, as previously indicated.

3. Quantum inverse scattering method

The essential motivation for the quantum inverse scattering method is the construction of a family of commuting matrices, known as transfer matrices. That is, we wish to construct an operator $t(u)$, where $u \in \mathbb{C}$ is called the spectral parameter, acting on some vector space, which represents the Hilbert space of physical states. Further we seek that

$$[t(u), t(v)] = 0 \qquad \forall u, v \in \mathbb{C}. \tag{8}$$

There are two significant consequences of (8). The first is that $t(u)$ may be diagonalized independently of u, that is the eigenvectors of $t(u)$ do not depend on u. This is the feature which makes the Bethe ansatz approach viable. Secondly, $t(u)$ commutes with all its derivatives, or more formally, taking the series expansion

$$t(u) = \sum_{k=-\infty}^{\infty} \mathfrak{C}_k u^k$$

it follows that

$$[\mathfrak{C}_k, \mathfrak{C}_j] = 0 \qquad \forall k, j.$$

Thus for any Hamiltonian which is expressible as a function of the operators $\mathfrak{C}_k$ only, each $\mathfrak{C}_k$ corresponds to an operator representing a constant of motion, since it will commute with the Hamiltonian. When the number of independent conserved quantities is equal to the number of degrees of freedom of the system, the model is said to be integrable.

Let V denote some fixed vector space of finite dimension n. The theory of exactly solvable quantum systems in this setting begins with an invertible operator, depending on the spectral parameter u,

$$R(u) \in \mathrm{End}(V \otimes V)$$

called the R-matrix. Here 'End' refers to the space of endomorphisms (square matrices), so $R(u)$ is effectively an $n^2 \times n^2$ matrix whose entries are scalar functions of u. From the R-matrix we define the Yang–Baxter algebra, denoted by Y, which is generated by the monodromy matrix $T(u)$, whose entries are elements of Y

$$R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v). \tag{9}$$

The above equation acts in the three-fold space $\mathrm{End}(V \otimes V) \otimes Y$ and the subscripts refer to the components of $\mathrm{End}(V \otimes V)$. In terms of the elementary matrices e^i_j, which have 1 in the (i, j) position and zeros elsewhere, we may write

$$R(u) = \sum_{i,j,k,l=1}^{n} R^{ik}_{jl}(u) e^i_j \otimes e^k_l \qquad T(u) = \sum_{i,j=1}^{n} e^i_j \otimes T^i_j(u).$$

Then

$$R_{12}(u) = \sum_{i,j,k,l=1}^{n} R^{ik}_{jl}(u) e^i_j \otimes e^k_l \otimes I$$

$$T_1(u) = \sum_{i,j=1}^{n} e^i_j \otimes I \otimes T^i_j(u)$$

$$T_2(u) = \sum_{i,j=1}^{n} I \otimes e^i_j \otimes T^i_j(u)$$

where I is the identity operator. In component form we may write

$$\sum_{j,l=1}^{n} R^{ik}_{jl}(u-v) T^j_p(u) T^l_q(v) = \sum_{j,l=1}^{n} T^k_j(v) T^i_l(u) R^{lj}_{pq}(u-v) \tag{10}$$

so the $R^{ik}_{jl}(u)$ give the structure constants of the algebra. Note that Y is actually an infinite-dimensional algebra, a basis for which $\left\{T^i_j[k]\right\}$ is obtained by taking the series expansions

$$T^i_j(u) = \sum_{k=-\infty}^{\infty} u^k T^i_j[k].$$

Imposing that Y is an associative algebra leads, through repeated use of (9), to the following equation in $\mathrm{End}(V \otimes V \otimes V) \otimes Y$:

$$\begin{aligned} T_1(u)T_2(v)T_3(w) &= (T_1(u)T_2(v))\, T_3(w) \\ &= R^{-1}_{12}(u-v)\, (T_2(v)T_1(u))\, T_3(w) R_{12}(u-v) \\ &= \cdots = R^{-1}_{12}(u-v) R^{-1}_{13}(u-w) R^{-1}_{23}(v-w) T_3(w) T_2(v) T_1(u) \\ &\quad \times R_{23}(v-w) R_{13}(u-w) R_{12}(u-v). \end{aligned} \tag{11}$$

Here $R_{jk}(u)$ denotes the matrix in $\mathrm{End}(V \otimes V \otimes V)$ acting non-trivially on the jth and kth spaces and as the identity on the remaining space. In a similar way one may deduce that

$$\begin{aligned} T_1(u)T_2(v)T_3(w) &= T_1(u)\, (T_2(v)T_3(w)) \\ &= R^{-1}_{23}(v-w) R^{-1}_{13}(u-w) R^{-1}_{12}(u-v) T_3(w) T_2(v) T_1(u) \\ &\quad \times R_{12}(u-v) R_{13}(u-w) R_{23}(v-w). \end{aligned} \tag{12}$$

A sufficient condition for (11) and (12) to be equivalent is that the R-matrix satisfies the *Yang–Baxter equation* acting in $\mathrm{End}(V \otimes V \otimes V)$:

$$R_{12}(u-v) R_{13}(u-w) R_{23}(v-w) = R_{23}(v-w) R_{13}(u-w) R_{12}(u-v). \tag{13}$$

The above shows that in this algebraic setting the Yang–Baxter equation arises as a natural way to impose associativity of the Yang–Baxter algebra Y. It also appears in many other contexts, such as classical two-dimensional statistical mechanics [7], knot theory [8, 73] and scattering theory [74].

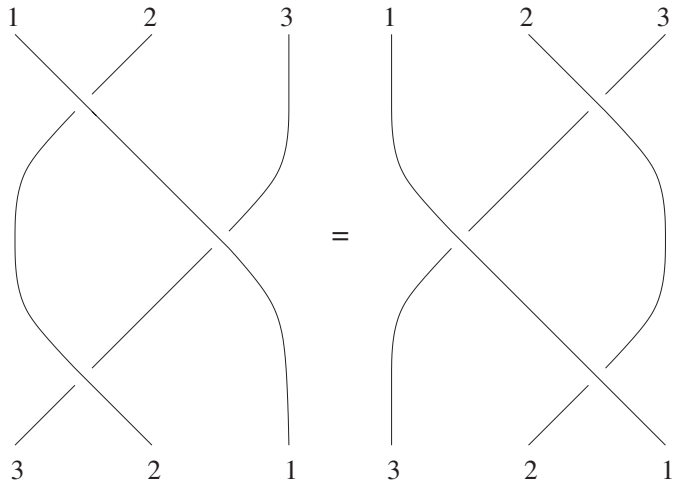

Figure 1. Graphical representation of the Yang–Baxter equation. In the context of knot theory, $R_{ij}(\infty)$ denotes the crossing of string i over string j. The Yang–Baxter equation states that the above two combinations of crossings are topologically equivalent. In scattering theory the matrix elements of $R_{ij}(u-v)$ give the amplitudes for the two-body scattering of particles labelled i and j, with rapidity variables u and v, respectively. The fact that the scattering depends only on the difference $u-v$ is a consequence of Lorentz invariance. The Yang–Baxter equation is a statement of equivalence for the two factorizations of three-body scattering in terms of two-body scattering. In classical two-dimensional statistical mechanics the matrix elements of $R_{ij}(u)$ give the allowed vertex weights at the lattice site labelled by (i, j). In this instance u can be parametrized in terms of the energy levels and temperature. The Yang–Baxter equation ensures the commutativity of the row-to-row transfer matrix, from which the partition function is constructed.

Here, we will only concern ourselves with the $gl(2)$ invariant R-matrix, which has the form [3, 6]

$$\begin{aligned} R(u) &= \frac{1}{u+\eta}(u \cdot I \otimes I + \eta P) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{14}$$

with $b(u) = u/(u+\eta)$, $c(u) = \eta/(u+\eta)$ and η is an arbitrary complex parameter. Above, P is the permutation operator which satisfies

$$P(x \otimes y) = y \otimes x \qquad \forall x, y \in V.$$

The R-matrix is $gl(2)$ invariant in that

$$[R(u), \mathfrak{g} \otimes \mathfrak{g}] = 0 \tag{15}$$

where $\mathfrak{g}$ is any 2×2 matrix.

For this case the Yang–Baxter algebra, denoted by $Y[gl(2)]$ has four elements

$$T(u) = \begin{pmatrix} T_1^1(u) & T_2^1(u) \\ T_1^2(u) & T_2^2(u) \end{pmatrix}. \tag{16}$$

For clarity and convenience we adopt the notation

$$A(u) = T_1^1(u) \qquad B(u) = T_2^1(u) \qquad C(u) = T_1^2(u) \qquad D(u) = T_2^2(u).$$

The full set of algebraic relations governed by (9) are

$$\begin{aligned}
&[A(u), A(v)] = [D(u), D(v)] = 0\\
&[B(u), B(v)] = [C(u), C(v)] = 0\\
&[A(u), D(v)] = \frac{\eta}{u-v}(C(v)B(u) - C(u)B(v))\\
&\qquad\qquad = \frac{\eta}{u-v}(B(u)C(v) - B(v)C(u))\\
&A(u)B(v) = \frac{u-v-\eta}{u-v}B(v)A(u) + \frac{\eta}{u-v}B(u)A(v)\\
&A(u)C(v) = \frac{u-v+\eta}{u-v}C(v)A(u) - \frac{\eta}{u-v}C(u)A(v)\\
&D(u)B(v) = \frac{u-v+\eta}{u-v}B(v)D(u) - \frac{\eta}{u-v}B(u)D(v)\\
&D(u)C(v) = \frac{u-v-\eta}{u-v}C(v)D(u) + \frac{\eta}{u-v}C(u)D(v)\\
&B(u)A(v) = \frac{u-v-\eta}{u-v}A(v)B(u) + \frac{\eta}{u-v}A(u)B(v)\\
&B(u)D(v) = \frac{u-v+\eta}{u-v}D(v)B(u) - \frac{\eta}{u-v}D(u)B(v)\\
&C(u)A(v) = \frac{u-v+\eta}{u-v}A(v)C(u) - \frac{\eta}{u-v}A(u)C(v)\\
&C(u)D(v) = \frac{u-v-\eta}{u-v}D(v)C(u) + \frac{\eta}{u-v}D(u)C(v)\\
&[B(u), C(v)] = \frac{\eta}{u-v}(A(u)D(v) - A(v)D(u))\\
&\qquad\qquad = \frac{\eta}{u-v}(D(v)A(u) - D(u)A(v)).
\end{aligned} \tag{17}$$

Next, suppose that we have a realization of $Y[gl(2)]$ acting on some vector space W, which we denote by $\pi : Y[gl(2)] \to \mathrm{End}\, W$. It is usual to refer to V as the auxiliary space and W as the physical space. Note that as $Y[gl(2)]$ is a quadratic algebra, any realization can be multiplied by an overall scaling factor and still satisfy relations (17). For later convenience we set

$$L(u) = \pi(T(u)) \in \mathrm{End}(V \otimes W)$$

which we refer to as an L-operator. Defining the transfer matrix through

$$t(u) = \pi\,(\mathrm{Tr}(T(u))) = \pi(A(u) + D(u)) \in \mathrm{End}\, W \tag{18}$$

it follows from (9) that the transfer matrices commute for different values of the spectral parameter, namely equation (8) is satisfied.

An important property of the Yang–Baxter algebra is that it has a co-multiplication structure which allows us to build tensor product realizations. In particular, given two L-operators $L^U \in \mathrm{End}(V \otimes U)$ and $L^W \in \mathrm{End}(V \otimes W)$, then $L = L^U L^W \in \mathrm{End}(V \otimes U \otimes W)$

is also an L-operator as can be seen from

$$\begin{aligned} R_{12}(u-v)L_1(u)L_2(v) &= R_{12}(u-v)L_1^U(u)L_1^W(u)L_2^U(v)L_2^W(v) \\ &= R_{12}(u-v)L_1^U(u)L_2^U(v)L_1^W(u)L_2^W(v) \\ &= L_2^U(v)L_1^U(u)R_{12}(u-v)L_1^W(u)L_2^W(v) \\ &= L_2^U(v)L_1^U(u)L_2^W(v)L_1^W(u)R_{12}(u-v) \\ &= L_2^U(v)L_2^W(v)L_1^U(u)L_1^W(u)R_{12}(u-v) \\ &= L_2(v)L_1(u)R_{12}(u-v). \end{aligned}$$

Furthermore, if $L(u)$ is an L-operator then so is $L(u+\alpha)$ for any α, since the R-matrix depends only on the difference of the spectral parameters. This property will prove important in all constructions given below.

3.1. The quasi-classical limit

The R-matrix (14) has the property

$$\lim_{\eta\to 0} R(u) = I \otimes I$$

which is known as the quasi-classical property. For any such R-matrix it is appropriate to write

$$R(u) = I \otimes I + \eta\mathcal{R}(u) + o(\eta^2) \qquad T_j^i(u) = \delta_j^i I + \eta\mathcal{T}_j^i(u) + o(\eta^2)$$

and substitute into (10). Equating the second-order terms in η yields the following relations:

$$\begin{aligned} \left[\mathcal{T}_p^i(u), \mathcal{T}_q^k(v)\right] = \sum_{j=1}^{n} \big(&\mathcal{R}_{pq}^{jk}(u-v)\mathcal{T}_j^i(u) - \mathcal{R}_{jq}^{ik}(u-v)\mathcal{T}_p^j(u) \\ &+ \mathcal{R}_{pq}^{ij}(u-v)\mathcal{T}_j^k(v) - \mathcal{R}_{pj}^{ik}(u-v)\mathcal{T}_q^j(v)\big) \end{aligned}$$

which we take to be the defining relations for the algebra denoted by $\mathcal{Y}$, to be called the Gaudin algebra. Gaudin used the quasi-classical limit to define classes of integrable spin chain Hamiltonians with long-range interactions [75, 76]. The algebraic approach which we follow here is due to Sklyanin [31, 77]. Observe that in the quasi-classical limit $\mathcal{Y}$ is an infinite-dimensional Lie algebra in contrast to the quadratic algebra structure of Y. Any realization of Y which admits the quasi-classical limit provides a realization of $\mathcal{Y}$.

For the case of the $gl(2)$ invariant R-matrix (14) let us write

$$\begin{aligned} A(u) &= I + \eta\mathcal{A}(u) + o(\eta^2) & B(u) &= \eta\mathcal{B}(u) + o(\eta^2) \\ C(u) &= \eta\mathcal{C}(u) + o(\eta^2) & D(u) &= I + \eta\mathcal{D}(u) + o(\eta^2). \end{aligned}$$

From (17) we determine that the full relations for the algebra $\mathcal{Y}[gl(2)]$ are

$$\begin{aligned} &[\mathcal{A}(u), \mathcal{A}(v)] = [\mathcal{B}(u), \mathcal{B}(v)] = 0 \\ &[\mathcal{C}(u), \mathcal{C}(v)] = [\mathcal{D}(u), \mathcal{D}(v)] = 0 \\ &[\mathcal{A}(u), \mathcal{D}(v)] = 0 \\ &[\mathcal{B}(u), \mathcal{C}(v)] = \frac{\mathcal{A}(u) - \mathcal{A}(v) + \mathcal{D}(v) - \mathcal{D}(u)}{u-v} \\ &[\mathcal{A}(u), \mathcal{B}(v)] = \frac{\mathcal{B}(u) - \mathcal{B}(v)}{u-v} \qquad [\mathcal{A}(u), \mathcal{C}(v)] = \frac{\mathcal{C}(v) - \mathcal{C}(u)}{u-v} \\ &[\mathcal{D}(u), \mathcal{B}(v)] = \frac{\mathcal{B}(v) - \mathcal{B}(u)}{u-v} \qquad [\mathcal{D}(u), \mathcal{C}(v)] = \frac{\mathcal{C}(u) - \mathcal{C}(v)}{u-v}. \end{aligned}$$

3.2. Examples of realization of the Yang–Baxter algebra

In order to construct a specific model, we must address the question of determining a realization of the Yang–Baxter algebra. Here we present several examples. The first realization comes from the R-matrix itself, since it is apparent by setting $w = 0$ in (13) that we can make the identification $L(u) = R(u)$ such that a realization of (9) is obtained. This realization satisfies the quasi-classical property, and is that used in the construction of the Heisenberg model [10–14]. A second realization is given by $L(u) = \mathfrak{g}$ (c-number realization), where $\mathfrak{g}$ is an arbitrary 2×2 matrix whose entries do not depend on u (although can depend on η). This follows from the fact that (15) holds for any such $\mathfrak{g}$.

There is a realization in terms of canonical boson operators $a, a^\dagger$ with the relation $[a, a^\dagger] = 1$ which reads [78]

$$L^a(u) = \begin{pmatrix} (1+\eta u)I + \eta^2 N & \eta a \\ \eta a^\dagger & I \end{pmatrix} \tag{19}$$

where $N = a^\dagger a$. There also exists a realization in terms of the $su(2)$ Lie algebra with generators S^z and $S^\pm$ [10–14],

$$L^S(u) = \frac{1}{u}\begin{pmatrix} uI + \eta S^z & \eta S^- \\ \eta S^+ & uI - \eta S^z \end{pmatrix} \tag{20}$$

subject to the commutation relations

$$[S^z, S^\pm] = \pm S^\pm \qquad [S^+, S^-] = 2S^z. \tag{21}$$

When the $su(2)$ algebra takes the spin-$1/2$ representation the resulting L-operator is equivalent to that given by the R-matrix. Another is realized in terms of the $su(1,1)$ generators K^z and $K^\pm$ [79, 80],

$$L^K(u) = \frac{1}{u}\begin{pmatrix} uI + \eta K^z & \eta K^- \\ -\eta K^+ & uI - \eta K^z \end{pmatrix} \tag{22}$$

with the commutation relations

$$[K^z, K^\pm] = \pm K^\pm \qquad [K^+, K^-] = -2K^z. \tag{23}$$

Each of the realizations $L^a(u)$, $L^S(u)$ and $L^K(u)$ satisfies the quasi-classical property, and thus affords a realization of the Gaudin algebra.

The discerning reader may note that $L^S(u)$ and $L^K(u)$ are in fact equivalent, which results from the Lie algebra homomorphism $\Upsilon : su(2) \to su(1,1)$ defined by

$$\Upsilon(S^z) = K^z \qquad \Upsilon(S^+) = -K^+ \qquad \Upsilon(S^-) = K^-$$

such that (21) is mapped to (23). For convenience we make the distinction between these two L-operators as the transformation Υ is non-unitary. This permits us to avoid the use of non-unitary realizations of the $su(2)$ algebra below. (Although, as will be seen, the realizations of the Yang–Baxter algebra may not be unitary.)

3.3. $\mathbb{Z}$-graded structure and $\mathbb{Z}$-graded realizations of the Yang–Baxter algebra

The Yang–Baxter algebra $Y[gl(2)]$ carries a $\mathbb{Z}$-graded structure that can be exploited in the construction of models of Bose–Einstein condensates, as explained in [66], which we now recount. We introduce an auxiliary operator Z, called the *grading operator*, satisfying the relations

$$[Z, X(u)] = p\{X(u)\} \cdot X(u) \tag{24}$$

where $X = A$, B, C or D and

$$p\{A(u)\} = p\{D(u)\} = 0 \qquad p\{B(u)\} = 1 \qquad p\{C(u)\} = -1.$$

We call $p\{X(u)\} \in \mathbb{Z}$ the *gradation* of $X(u)$, and extend the gradation operation to the entire algebra by the requirement

$$p\{\theta \cdot \phi\} = p\{\theta\} + p\{\phi\} \qquad \forall \theta, \phi \in \mathcal{A}.$$

This definition for the grading operator is consistent with the defining relations (17).

Let us now define a class of realizations of $Y[gl(2)]$ which we call $\mathbb{Z}$-graded realizations. We say that a vector space W, equipped with an endomorphism z, is a $\mathbb{Z}$-graded vector space, denoted by (W, z), if it admits a decomposition into subspaces

$$W = \bigoplus_{j=-\infty}^{\infty} W_j$$

such that

$$zW_j = j \cdot W_j \qquad j \in \mathbb{Z}.$$

Note that some of the W_j may be trivial subspaces. Formally, the grading operator can be used to define the following projection operators:

$$P_j = \prod_{\substack{k=-\infty \\ k \neq j}}^{\infty} \frac{(z - kI)}{(j - k)} \tag{25}$$

such that

$$P_k P_j = \delta_{kj} P_j \qquad P_k W_j = \delta_{kj} W_k.$$

We say that a $\mathbb{Z}$-graded vector space

$$W' = \bigoplus_{j=-\infty}^{\infty} W'_j$$

is *equivalent* to W if for some $k \in \mathbb{Z}$ there exists a vector space isomorphism between W'_j and W_{j+k} for all j. This terminology is motivated by the fact that for a given (W, z) one can always generate another $\mathbb{Z}$-graded space (W', z') through the mappings $W'_j \to W_{j+k}$, $z' \to z - kI$ for any $k \in \mathbb{Z}$.

For a given $\mathbb{Z}$-graded W we say that $\pi : Y[gl(2)] \to \operatorname{End} W$ provides a $\mathbb{Z}$-graded realization of $Y[gl(2)]$ if $\pi(Z) = z$ and relations (17) and (24) are preserved. In such a case we can write

$$\pi(X(u)) = \sum_{j=-\infty}^{\infty} X(u, j)$$

and the matrices $X(u, j)$ satisfy

$$X(u, j)W_k = 0 \qquad \text{for} \quad j \neq k.$$

More specifically, this means that for $|\psi_j\rangle \in W_j$ we have

$$\pi(X(u)Y(v))|\psi_j\rangle = X(u, j + p\{Y(u)\})Y(v, j)|\psi_j\rangle.$$

In view of the equivalence of $\mathbb{Z}$-graded vector spaces defined above, there can also exist equivalent realizations. We can define a realization π' equivalent to π by specifying some $k \in \mathbb{Z}$ such that

$$\pi'(Z) = \pi(Z - kI)$$

and for

$$\pi'(X(u)) = \sum_{j=-\infty}^{\infty} X'(u, j)$$

the matrices $X'(u, j)$ are defined by

$$X'(u, j) = X(u, j + k) \qquad \forall j \in \mathbb{Z}.$$

For the $\mathbb{Z}$-graded case we may express the transfer matrix as

$$t(u) = \sum_{j=-\infty}^{\infty} t(u, j)$$

such that

$$t(u, j)W_k = 0 \qquad \text{for} \quad j \neq k$$

and

$$[t(u, j), t(v, k)] = 0 \qquad \forall j, k \in \mathbb{Z} \quad u, v \in \mathbb{C}.$$

Since $p\{t(u)\} = 0$, the diagonalization of $t(u)$ is thus reduced to the diagonalization of each of the matrices $t(u, j)$ on the $\mathbb{Z}$-graded component W_j, where we have

$$[t(u, j), t(v, j)] = 0 \qquad \forall u, v \in \mathbb{C}.$$

We may restrict our attention to the case of $t(u, 0)$, as each $t(u, j)$ is equivalent to some $t'(u, 0)$ through the use of equivalent realizations as introduced above.

3.4. Examples of $\mathbb{Z}$-graded realizations

Next we give two non-trivial $\mathbb{Z}$-graded realizations of the algebra $Y[gl(2)]$. One is expressible in terms of two Heisenberg algebras with generators $a_i, a_i^\dagger, i = 1, 2$, and reads $X(u, j) = \tilde{X}(u, j)P_j$ with

$$\begin{aligned}
\tilde{A}(u, j)) &= u^2 + \eta u N + \eta^2 N_1 N_2 - \eta(N_1 - N_2)\omega(N + jI) - \omega^2(N + jI) + a_2^\dagger a_1 \\
\tilde{B}(u, j) &= (u + \omega(N + jI) + \eta N_1)a_2 + \eta^{-1} a_1 \\
\tilde{C}(u, j) &= a_1^\dagger(u - \omega(N + jI) + \eta N_2) + \eta^{-1} a_2^\dagger \\
\tilde{D}(u, j) &= a_1^\dagger a_2 + \eta^{-2} \\
Z &= k \cdot I - N.
\end{aligned} \tag{26}$$

Above, k is an arbitrary scalar, P_j are the projections defined by (25), $N_i = a_i^\dagger a_i$, $N = N_1 + N_2$ and $\omega(x)$ is an arbitrary polynomial function of x. The operators act on the Fock space spanned by the basis vectors given by (2). Note that in the case when $\omega(x)$ is constant, the above realization reduces to that discussed in [49, 50, 62–64] and is factorizable into two local realizations of the Yang–Baxter algebra expressible in terms of the two Heisenberg algebras, namely

$$L(u) = \eta^{-2} L_1^a(u - \eta^{-1} + \omega) L_2^a(u - \eta^{-1} - \omega).$$

It is important to note that for generic $\omega(x)$ no such factorization exists.

Another $\mathbb{Z}$-graded realization of the Yang–Baxter algebra is $X(u, j) = \tilde{X}(u, j)P_j$ with

$$\begin{aligned}
\tilde{A}(u, j) &= -\eta u^2 + u(1 - \eta^2(K^z + N_c) - \eta\omega(K^z + N_c + jI)) \\
&\quad + \eta K^z - \eta^2 K^z \omega(K^z + N_c + jI) - \eta^3 N_c K^z + \eta^2 c K^+ \\
\tilde{B}(u, j) &= \eta(1 - \eta u - \eta\omega(K^z + N_c + jI) - \eta^2 N_c)K^- - \eta c(u - \eta K^z) \\
\tilde{C}(u, j) &= \eta c^\dagger(u + \eta K^z) - \eta K^+ \\
\tilde{D}(u, j) &= u - \eta K^z + \eta^2 c^\dagger K^- \\
Z &= k \cdot I - K^z - N_c + \kappa.
\end{aligned} \tag{27}$$

Above, k is again arbitrary, the operators $c, c^\dagger$ form a Heisenberg algebra, with $N_c = c^\dagger c$, and as before the operators K^z, K^+, K^- satisfy the $su(1, 1)$ relations (23). It is assumed that the $su(1, 1)$ operators are realized in terms of irreducible representations of lowest weight κ. As in the previous example, $\omega(x)$ is an arbitrary polynomial function of x and the above realization is factorizable only in the case when $\omega(x)$ is constant. In this instance we have

$$L(u) = u\mathfrak{g}L^a(u - \eta^{-1} + \omega)L^K(u)$$

where $\mathfrak{g} = \mathrm{diag}(-1, 1)$.

4. Algebraic Bethe ansatz method of solution

A key step in successfully applying the algebraic Bethe ansatz approach is finding a suitable pseudovacuum state, $|\chi\rangle$, which has the properties

$$A(u)|0\rangle = a(u)|\chi\rangle \qquad B(u)|0\rangle = 0 \qquad C(u)|0\rangle \neq 0 \qquad D(u)|0\rangle = d(u)|\chi\rangle$$

where $a(u)$ and $d(u)$ are scalar functions. Note that for ease of notation throughout we will omit the symbol π denoting the realization of $Y[gl(2)]$. Next choose the Bethe state

$$|\vec{v}\rangle \equiv |v_1, \ldots, v_M\rangle = \prod_{i=1}^{M} C(v_i)|\chi\rangle. \tag{28}$$

Note that because $[C(u), C(v)] = 0$, the ordering is not important in the product of (28). The approach of the algebraic Bethe ansatz is to use relations (17) to determine the action of $t(u)$ on $|\vec{v}\rangle$. First let us consider the action of $A(u)$ on $|\vec{v}\rangle$, namely

$$A(u)|\vec{v}\rangle = A(u)C(u_i)|\vec{v}_i\rangle$$

where

$$|\vec{v}_i\rangle \equiv |v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_M\rangle.$$

Now

$$\begin{aligned}
A(u)|\vec{v}\rangle &= \frac{u - v_i + \eta}{u - v_i} C(v_i)A(u)|\vec{v}_i\rangle - \frac{\eta}{u - v_i} C(u)A(v_i)|\vec{v}_i\rangle \\
&= \frac{u - v_i + \eta}{u - v_i} C(v_i)A(u)C(v_j)|\vec{v}_{ij}\rangle - \frac{\eta}{u - v_i} C(u)A(v_i)C(v_j)|\vec{v}_{ij}\rangle \\
&= \left(\frac{u - v_i + \eta}{u - v_i}\right)\left(\frac{u - v_j + \eta}{u - v_j}\right) C(v_i)C(v_j)A(u)|\vec{v}_{ij}\rangle \\
&\quad - \left(\frac{u - v_i + \eta}{u - v_i}\right)\left(\frac{\eta}{u - v_j}\right) C(v_i)C(u)A(v_j)|\vec{v}_{ij}\rangle \\
&\quad - \left(\frac{\eta}{u - v_i}\right)\left(\frac{v_i - v_j + \eta}{v_i - v_j}\right) C(u)C(v_j)A(v_i)|\vec{v}_{ij}\rangle \\
&\quad + \left(\frac{\eta}{u - v_i}\right)\left(\frac{\eta}{v_i - v_j}\right) C(u)C(v_i)A(v_j)|\vec{v}_{ij}\rangle
\end{aligned} \tag{29}$$

where

$$|\vec{v}_{ij}\rangle \equiv |v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_M\rangle.$$

Proceeding further we find the general form

$$\begin{aligned} A(u)|\vec{v}\rangle = &\left(\prod_{i=1}^{M} \frac{u - v_i + \eta}{u - v_i}\right)\left(\prod_{i=1}^{M} C(v_i)\right) A(u)|\chi\rangle \\ &- \frac{\eta}{u - v_i}\left(\prod_{j\neq i}^{M} \frac{v_i - v_j + \eta}{v_i - v_j}\right) C(u)\left(\prod_{k\neq i}^{M} C(v_k)\right) A(v_i)|\chi\rangle \\ &+ \text{other linearly independent terms} \\ = &\, a(u)\left(\prod_{i=1}^{M} \frac{u - v_i + \eta}{u - v_i}\right)|\vec{v}\rangle - \frac{\eta a(v_i)}{u - v_i}\left(\prod_{j\neq i}^{M} \frac{v_i - v_j + \eta}{v_i - v_j}\right) C(u)|\vec{v}_i\rangle \\ &+ \text{other linearly independent terms.} \end{aligned}$$

Above, each of the other linearly independent terms is a vector of the form $C(u)|\vec{v}_j\rangle$, $j \neq i$, multiplied by some scalar. There are no other possibilities. To determine what the coefficients are we note that the above equation is valid for any choice of i. Hence, we conclude that

$$A(u)|\vec{v}\rangle = a(u)\left(\prod_{i=1}^{M} \frac{u - v_i + \eta}{u - v_i}\right)|\vec{v}\rangle - \sum_{i=1}^{M} \frac{\eta a(v_i)}{u - v_i}\left(\prod_{j\neq i}^{M} \frac{v_i - v_j + \eta}{v_i - v_j}\right) C(u)|\vec{v}_i\rangle.$$

The coefficients of the terms $C(u)|\vec{v}_i\rangle$ are called *unwanted terms* for reasons that will soon become apparent.

We now perform the same procedure for $D(u)$,

$$\begin{aligned} D(u)|\vec{v}\rangle &= D(u)C(v_i)|\vec{v}_i\rangle \\ &= \left(\frac{u - v_i - \eta}{u - v_i}\right) C(v_i)D(u)|\vec{v}_i\rangle + \frac{\eta}{u - v_i} C(u)D(v_i)|\vec{v}_i\rangle \\ &= d(u)\left(\prod_{i=1}^{M} \frac{u - v_i - \eta}{u - v_i}\right)|\vec{v}\rangle + \sum_{i=1}^{M} \frac{\eta d(v_i)}{u - v_i}\left(\prod_{j\neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j}\right) C(u)|\vec{v}_i\rangle. \end{aligned} \tag{30}$$

The final result for the action of the transfer matrix is

$$\begin{aligned} t(u)|\vec{v}\rangle = &(A(u) + D(u))|\vec{v}\rangle \\ = &\, \Lambda(u, \vec{v})|\vec{v}\rangle - \sum_{i}^{M} \frac{\eta a(v_i)}{u - v_i}\left(\prod_{j\neq i}^{M} \frac{v_i - v_j + \eta}{v_i - v_j}\right) C(u)|\vec{v}_i\rangle \\ &+ \sum_{i=1}^{M} \frac{\eta d(v_i)}{u - v_i}\left(\prod_{j\neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j}\right) C(u)|\vec{v}_i\rangle \end{aligned} \tag{31}$$

where

$$\Lambda(u, \vec{v}) = a(u)\prod_{i=1}^{M} \frac{u - v_i + \eta}{u - v_i} + d(u)\prod_{i=1}^{M} \frac{u - v_i - \eta}{u - v_i}. \tag{32}$$

The above shows that $|\vec{v}\rangle$ becomes an eigenstate of the transfer matrix with eigenvalue (32) whenever the unwanted terms cancel. This occurs when the *Bethe ansatz equations*

$$\frac{a(v_i)}{d(v_i)} = \prod_{j\neq i}^{M} \frac{v_i - v_j - \eta}{v_i - v_j + \eta} \qquad i = 1, \ldots, M. \tag{33}$$

are satisfied. Throughout we adopt the notation

$$\{v_i\} \equiv \{v_1, v_2, \ldots, v_M\}$$

for such a solution.

Note that in the derivation of the Bethe ansatz equations it is required that $v_i \neq v_j \; \forall i, j$. This is a result of the Pauli principle for Bethe ansatz solvable models, as developed in [81] for the one-dimensional Bose gas with delta-function interactions. For generic functions $a(u), d(u)$, essentially the same argument as [81] can be applied to draw the same conclusion. To give an indication why this is the case, consider (29) in the limit $v_i \to v_j$

$$A(u)|\vec{v}\rangle = \left(\frac{u - v_i + \eta}{u - v_i}\right)^2 C(v_i)^2 A(u)|\vec{v}_{ij}\rangle - \left(\frac{u - v_i + \eta}{u - v_i}\right)\left(\frac{\eta}{u - v_i}\right) C(v_i)C(u)A(v_i)|\vec{v}_{ij}\rangle + \left(\frac{\eta^2}{u - v_i}\right) C(u)\left[\frac{\mathrm{d}}{\mathrm{d}w}C(v_i)\cdot A(v_i) - C(v_i)\cdot\frac{\mathrm{d}}{\mathrm{d}w}A(v_i)\right]|\vec{v}_{ij}\rangle.$$

This equation shows that new types of unwanted terms occur which depend on the derivatives of the elements of $Y[gl(2)]$, and this leads to an overdetermined system of equations which do not admit a solution. Another viewpoint is to note that (up to an overall scaling factor, and under suitable assumptions for the forms of $a(u)$ and $d(u)$) the eigenvalues $\Lambda(u, \vec{v})$ are analytic functions of u. Assuming that the poles in (32) are simple then the Bethe ansatz equations (33) are equivalent to the statement that the residue vanishes at each pole, i.e.,

$$\lim_{u \to v_i}(u - v_i)\Lambda(u, \vec{v}) = 0$$

leads to (33). For non-simple poles however there are additional Bethe ansatz equations, which cannot be satisfied. To illustrate this, we consider the simplest case where, for all values of η, all poles are simple except for one, say at $u = v_j$, which is of second order. In such a case, let us write

$$\Lambda(u, \vec{v}) = a(u)\frac{(u - v_j + \eta)^2}{(u - v_j)^2}\prod_{k \neq j}^{M}\frac{(u - v_k + \eta)}{(u - v_k)} + d(u)\frac{(u - v_j - \eta)^2}{(u - v_j)^2}\prod_{k \neq j}^{M}\frac{(u - v_k - \eta)}{(u - v_k)}.$$

Analyticity of $\Lambda(u, \vec{v})$ requires

$$0 = \lim_{u \to v_j}(u - v_j)^2\Lambda(u, \vec{v})$$

leading to the Bethe ansatz equations

$$\frac{a(v_j)}{d(v_j)} = -\prod_{k \neq j}^{M}\frac{v_j - v_k - \eta}{v_j - v_k + \eta}.$$

In particular note that

$$\lim_{\eta \to 0}\frac{a(v_j)}{d(v_j)} = -1. \tag{34}$$

Furthermore

$$\begin{aligned} 0 &= \lim_{u \to v_j}(u - v_j)\Lambda(u, \vec{v}) \\ &= \lim_{u \to v_j}\frac{\mathrm{d}}{\mathrm{d}u}[(u - v_j)^2\Lambda(u, \vec{v})] \\ &= \lim_{u \to v_j}\left(a'(u)(u - v_j + \eta)^2\prod_{k \neq j}^{M}(u - v_k + \eta) + 2a(u)\prod_{k=1}^{M}(u - v_k + \eta)\right. \end{aligned}$$

$$
\begin{aligned}
&+a(u)(u-v_j+\eta)^2\sum_{l\neq j}^{M}\prod_{k\neq l,j}^{M}(u-v_k+\eta)\\
&+d'(u)(u-v_j-\eta)^2\prod_{k\neq j}^{M}(u-v_k-\eta)+2d(u)\prod_{k=1}^{M}(u-v_k-\eta)\\
&+d(u)(u-v_j-\eta)^2\sum_{l\neq j}^{M}\prod_{k\neq l,j}^{M}(u-v_k-\eta)\Bigg)\\
&=2\eta a(v_j)\prod_{k\neq j}^{M}(v_j-v_k+\eta)-2\eta d(v_j)\prod_{k\neq j}^{M}(v_j-v_k-\eta)+o(\eta^2).
\end{aligned}
$$

The above implies that

$$\lim_{\eta\to 0}\frac{a(v_j)}{d(v_j)}=1$$

in contradiction to (34), proving the claim that one cannot have a single second-order pole. The extension to more complicated non-simple pole structures, though tedious, is straightforward.

Finally, the following identity is useful and is easily derived from the Bethe ansatz equations:

$$\prod_{i=1}^{M}a(v_i)=\prod_{i=1}^{M}d(v_i). \tag{35}$$

4.1. Extension to $\mathbb{Z}$-graded realizations

In order to formulate the algebraic Bethe ansatz solution for the class of $\mathbb{Z}$-graded realizations, we begin with the observation from (17) that the following relations hold (amongst others):

$$
\begin{aligned}
&[A(u,j),A(v,j)]=[D(u,j),D(v,j)]=0\\
&B(u,j)B(v,j-1)=B(v,j)B(u,j-1)\\
&C(u,j)C(v,j+1)=C(v,j)C(u,j+1)\\
&A(u,j)C(v,j+1)=\frac{u-v+\eta}{u-v}C(v,j+1)A(u,j+1)-\frac{\eta}{u-v}C(u,j+1)A(v,j+1)\\
&D(u,j)C(v,j+1)=\frac{u-v-\eta}{u-v}C(v,j+1)D(u,j+1)+\frac{\eta}{u-v}C(u,j+1)D(v,j+1).
\end{aligned}
\tag{36}
$$

Again, we assume the existence of a pseudovacuum vector $|\chi\rangle\in W_k$ such that

$$
\begin{aligned}
&A(u,k)|\chi\rangle=a(u,k)|\chi\rangle \qquad && B(u,k)|\chi\rangle=0\\
&C(u,k)|\chi\rangle\neq 0 && D(u,k)|\chi\rangle=d(u,k)|\chi\rangle.
\end{aligned}
$$

In particular, for the realization (26) the Fock vacuum serves as the pseudovacuum. In the case of (27) we choose the pseudovacuum to be the tensor product of the Fock vacuum with the $su(1,1)$ lowest weight state of weight κ.

The above implies that $|\chi\rangle$ is a maximal weight vector with respect to Z. Without loss of generality we can choose $k=M$, due to the equivalence of realizations discussed earlier, and look for Bethe states defined by

$$|\vec{v}\rangle=C(v_1,1)C(v_2,2)\cdots C(v_M,M)|\chi\rangle. \tag{37}$$

It is easy to check that the above Bethe state is symmetric with respect to interchange of the variables v_i, a feature which plays a crucial role below. In particular, this means that we may write

$$\begin{aligned}|\vec{v}\rangle &= C(v_i, 1)|\vec{v}_i\rangle \\ &= C(v_i, 1)C(v_j, 2)|\vec{v}_{ij}\rangle \\ &= C(v_j, 1)C(v_i, 2)|\vec{v}_{ij}\rangle\end{aligned}$$

where now

$$|\vec{v}_i\rangle = C(v_1, 2)\cdots C(v_{i-1}, i)C(v_{i+1}, i+1)\cdots C(v_M, M)|\chi\rangle$$

$$\begin{aligned}|\vec{v}_{ij}\rangle = {} & C(v_1, 3)\cdots C(v_{i-1}, i+1)C(v_{i+1}, i+2) \\ & \times \cdots C(v_{j-1}, j)C(v_{j+1}, j+1)\cdots C(v_M, M)|\chi\rangle.\end{aligned}$$

Acting $A(u, 0)$ and $D(u, 0)$ on the Bethe state we have, by following the general procedure detailed above,

$$A(u,0)|\vec{v}\rangle = a(u,M)\left(\prod_{i=1}^{M}\frac{u-v_i+\eta}{u-v_i}\right)|\vec{v}\rangle - \sum_{i=1}^{M}\frac{\eta a(v_i,M)}{u-v_i}\left(\prod_{j\neq i}^{M}\frac{v_i-v_j+\eta}{v_i-v_j}\right)C(u,1)|\vec{v}_i\rangle$$

$$D(u,0)|\vec{v}\rangle = d(u,M)\left(\prod_{i=1}^{M}\frac{u-v_i-\eta}{u-v_i}\right)|\vec{v}\rangle + \sum_{i=1}^{M}\frac{\eta d(v_i,M)}{u-v_i}\left(\prod_{j\neq i}^{M}\frac{v_i-v_j-\eta}{v_i-v_j}\right)C(u,1)|\vec{v}_i\rangle.$$

Requiring $|\vec{v}\rangle$ to be an eigenstate of $t(u, 0)$ leads to the Bethe ansatz equations

$$\frac{a(v_i,M)}{d(v_i,M)} = \prod_{j\neq i}^{M}\frac{v_i-v_j-\eta}{v_i-v_j+\eta} \qquad i = 1,\ldots,M \tag{38}$$

and the corresponding eigenvalue of the matrix $t(u, 0)$ is

$$\Lambda(u,0,\vec{v}) = a(u,M)\prod_{i=1}^{M}\frac{u-v_i+\eta}{u-v_i} + d(u,M)\prod_{i=1}^{M}\frac{u-v_i-\eta}{u-v_i}. \tag{39}$$

5. Scalar products of states

Recall that in the usual algebraic Bethe ansatz for the algebra $Y[gl(2)]$ there is a formula originally due to Slavnov [32] (see also [13, 29]) for the wavefunction scalar products, which is

$$\begin{aligned}S(\vec{w}:\vec{v}) &= \{\vec{w}|\vec{v}\rangle \\ &= \{\vec{v}|\vec{w}\rangle \\ &= \frac{\det F}{\prod_{k>l}^{M}(v_k-v_l)\prod_{i<j}^{M}(w_i-w_j)}\end{aligned} \tag{40}$$

with the entries of the $M \times M$ matrix F given by

$$F_{ij} = \frac{\eta d(w_i)}{(v_j-w_i)}\left(a(v_j)\prod_{k\neq i}^{M}(v_j-w_k+\eta) - d(v_j)\prod_{k\neq i}^{M}(v_j-w_k-\eta)\right) \tag{41}$$

and $\{\vec{u}|$ is the *left* vector defined by

$$\{\vec{u}| = \langle\chi|B(u_M)\cdots B(u_1)$$

for any choice of $\{u_i\}$. Above, $\{w_i\}$ provide a solution to the Bethe ansatz equations (38) and the parameters $\{v_j\}$ are arbitrary. In using the Slavnov formula it is assumed that the pseudovacuum state has norm equal to 1. Defining

$$G = F \cdot \Gamma$$

where Γ is a diagonal matrix with entries

$$\Gamma_{ij} = \delta_{ij} \frac{(v_j - w_j)}{\prod_{k=1}^{M}(v_j - w_k)}$$

the Slavnov formula may be expressed in the equivalent form

$$S(\vec{w} : \vec{v}) = \frac{\prod_{p=1}^{M} \prod_{q \neq p}^{M} (v_p - w_q)}{\prod_{k>l}^{M}(v_k - v_l) \prod_{i<j}^{M}(w_i - w_j)} \det G \tag{42}$$

with

$$G_{ij} = \frac{\eta d(w_i)(v_j - w_j)}{(v_j - w_i)^2} \left(a(v_j) \prod_{k \neq i}^{M} \frac{(v_j - w_k + \eta)}{(v_j - w_k)} - d(v_j) \prod_{k \neq i}^{M} \frac{(v_j - w_k - \eta)}{(v_j - w_k)} \right). \tag{43}$$

We will find it convenient to use both forms (40) and (42) of the Slavnov formula.

The Yang–Baxter algebra $Y[gl(2)]$ admits a conjugation operation $\dagger : Y[gl(2)] \to Y[gl(2)]$ defined by

$$A(u)^\dagger = A(u^*) \qquad B(u)^\dagger = C(u^*) \qquad C(u)^\dagger = B(u^*) \qquad D(u)^\dagger = D(u^*)$$

and extended to all of $Y[gl(2)]$ through

$$(\theta \cdot \phi)^\dagger = \phi^\dagger \cdot \theta^\dagger \qquad \forall \theta, \phi \in Y[gl(2)]$$

such that the defining relations (17) are preserved. Above, $*$ is used to denote complex conjugation. Consequently the right vector

$$\begin{aligned} |\vec{v}\} &= \{\vec{v}|^\dagger \\ &= B(v_1^*)^\dagger \cdots B(v_M^*)^\dagger |\chi\rangle \end{aligned}$$

is also an eigenvector of the transfer matrix whenever the Bethe ansatz equations for the parameters $\{v_i\}$ are satisfied. However, it is apparent that the $\mathbb{Z}$-graded realizations (26) and (27) we have introduced are not unitary, and generally

$$\langle \vec{v}| = |\vec{v}\rangle^\dagger \neq \{\vec{v}|.$$

On the other hand, numerical analysis for the models (1), (3), (4) and (5) indicates that for fixed particle numbers, and generic values of the coupling parameters, the energy spectrum is free of degeneracies. This is presumably due to the fact that the only Lie algebra symmetries for these models are $u(1)$ invariances corresponding to conservation of particle numbers, and the non-degenerate spectra are examples of Hund's non-crossing rule [82, 83]. We also assert that for a given $\{v_i\}$, satisfying the Bethe ansatz equations (59), this set of parameters is equivalent to $\{v_i^*\}$, i.e., $v_i^* = v_j$ for some $j = 1, \ldots, M$. It is clear that for $\{v_i\}$ satisfying (33), so does $\{v_i^*\}$ (in all our examples, as the Hamiltonians are real, the functions $a(u)$ and $d(u)$ are real, as will be seen below). Since the eigenvalues of the Hamiltonian are real, we have

$$E(\vec{v}) = E^*(\vec{v}) = E(\vec{v}^*).$$

Under the belief that the spectrum is multiplicity free, we then deduce $\{v_i\} \sim \{v_i^*\}$. Whenever this is the case, we can conclude that the eigenvectors are real and

$$\langle \vec{v}| = \zeta(\vec{v})\{\vec{v}| \tag{44}$$

for some non-zero real-valued scalar $\zeta(\vec{v})$. Thus the Slavnov formula can be invoked for the computation of form factors. Throughout we will always assume (44) to be the case, which implies that

$$\{\vec{w}|\vec{v}\rangle = 0 \qquad \text{for} \quad \vec{v} \neq \vec{w} \tag{45}$$

whenever $\{v_i\}$ and $\{w_j\}$ both satisfy the Bethe ansatz equations. This result (45) can be proved directly, independent of (44), as shown in [30]. Note when $\vec{w} = \vec{v}$ we need to take a limit for the diagonal entries of F to compute the square of the norm. This yields

$$\begin{aligned} F_{ii} = {} & d(v_i)\left(a'(v_i)\prod_{k=1}^{M}(v_i - v_k + \eta) - d'(v_i)\prod_{k=1}^{M}(v_i - v_k - \eta)\right) \\ & + d(v_i)\left(\sum_{l\neq i}^{M}\frac{a(v_i)}{v_i - v_l + \eta}\prod_{k=1}^{M}(v_i - v_k + \eta) - \sum_{l\neq i}^{M}\frac{d(v_i)}{v_i - v_l - \eta}\prod_{k=1}^{M}(v_i - v_k - \eta)\right) \end{aligned} \tag{46}$$

where the prime denotes the derivative.

From the Slavnov formula the matrix elements of the operators $B(u)$, $C(u)$ follow directly, as the set of parameters $\{v_i\}$ are arbitrary. We will now derive an expression for the form factors of the operator $D(u)$, which will prove useful for later calculations. From expression (30) for the action of $D(u)$ on an arbitrary Bethe vector we may deduce that for both $\{v_i\}$ and $\{w_j\}$ satisfying the Bethe ansatz equations

$$\begin{aligned} \{\vec{w}|D(u)|\vec{v}\rangle &= d(u)\left(\prod_{p=1}^{M}\frac{u - v_p - \eta}{u - v_p}\right)\{\vec{w}|\vec{v}\rangle + \sum_{q=1}^{M}\frac{\eta d(v_q)}{u - v_q}\left(\prod_{j\neq q}^{M}\frac{v_q - v_j - \eta}{v_q - v_j}\right)\{\vec{w}|C(u)|\vec{v}_i\rangle \\ &= d(u)\left(\prod_{p=1}^{M}\frac{u - w_p - \eta}{u - w_p}\right)\left[S(\vec{w}:\vec{v}) + \sum_{q=1}^{M}\frac{\eta d(v_q)}{(u - v_q)d(u)}\left(\prod_{k=1}^{M}\frac{u - w_k}{u - w_k - \eta}\right)\right. \\ &\quad \left.\times\left(\prod_{j\neq q}^{M}\frac{v_q - v_j - \eta}{v_q - v_j}\right)S(\vec{w}:v_1,\ldots,v_{q-1},u,v_{q+1},\ldots,v_M)\right] \\ &= \frac{d(u)}{\prod_{k>l}^{M}(v_k - v_l)\prod_{i<j}^{M}(w_i - w_j)}\left(\prod_{p=1}^{M}\frac{u - w_p - \eta}{u - w_p}\right) \\ &\quad \times\left(\det F + \Theta(u)\sum_{q=1}^{M}\det F^{(q)}\right) \end{aligned}$$

where the matrices $F^{(q)}$ are defined by

$$\begin{aligned} F^{(q)}_{pl} &= F_{pl} \qquad \text{for} \quad l \neq q \\ F^{(q)}_{pq} &= Q_{pq} \end{aligned} \tag{47}$$

Q is the rank-1 matrix with elements

$$Q_{ij} = \frac{\eta d(w_i)d(v_j)}{(u - w_i)(u - w_i - \eta)}\left(1 - \frac{a(u)}{d(u)}\left(\prod_{k\neq i}^{M}\frac{u - w_k + \eta}{u - w_k - \eta}\right)\right)\prod_{l=1}^{M}(v_j - v_l - \eta)$$

and

$$\Theta(u) = \left(\prod_{k=1}^{M} \frac{u - w_k}{u - v_k} \right). \tag{48}$$

Using the fact that for $\vec{v} \neq \vec{w}$, $\det F = 0$, whereas for $\vec{v} = \vec{w}$, $\Theta(u) = 1$, allows us to write

$$\begin{aligned} \{\vec{w}|D(u)|\vec{v}\rangle &= \frac{\Theta(u)d(u)}{\prod_{k>l}^{M}(v_k - v_l)\prod_{i<j}^{M}(w_i - w_j)} \left(\prod_{i=1}^{M} \frac{u - w_i - \eta}{u - w_i} \right) \left(\det F + \sum_{j=1}^{M} \det F^{(j)} \right) \\ &= \frac{\Theta(u)d(u)}{\prod_{k>l}^{M}(v_k - v_l)\prod_{i<j}^{M}(w_i - w_j)} \left(\prod_{i=1}^{M} \frac{u - w_i - \eta}{u - w_i} \right) \det(F + Q). \end{aligned} \tag{49}$$

The last line above follows from the fact that if X is any $M \times M$ matrix and Y is any rank-1 $M \times M$ matrix then

$$\det(X + Y) = \det X + \sum_{j=1}^{M} \det X^{(j)}$$

where

$$\begin{aligned} X_{ij}^{(j)} &= Y_{ij} \\ X_{kl}^{(j)} &= X_{kl} \qquad \text{for} \quad j \neq l. \end{aligned}$$

A similar result can be derived for the form factors of $A(u)$. Alternatively, one can obtain them from (32) and (49) through

$$\{\vec{w}|A(u)|\vec{v}\rangle = \Lambda(u, \vec{v})\{\vec{w}|\vec{v}\rangle - \{\vec{w}|D(u)|\vec{v}\rangle. \tag{50}$$

The Slavnov formula can be extended to include generic $\mathbb{Z}$-graded realizations as noted in [66]. This is achieved by simply replacing $a(u)$ and $d(u)$ with $a(u, M)$ and $d(u, M)$ in (41) and (43). All the results derived above also extend analogously to the $\mathbb{Z}$-graded case.

5.1. The quasi-classical limit

Assuming that the quasi-classical limit exists and in particular

$$a(u) = 1 + \eta \mathsf{a}(u) + o(\eta^2) \qquad d(u) = 1 + \eta \mathsf{d}(u) + o(\eta^2)$$

it is straightforward to obtain the Slavnov formula in the quasi-classical limit. We obtain directly from (40) and (42)

$$\begin{aligned} \mathcal{S}(\vec{w} : \vec{v}) &= \langle \chi | \mathcal{B}(w_M) \cdots \mathcal{B}(w_1) \mathcal{C}(v_1) \cdots \mathcal{C}(v_M) | \chi \rangle \\ &= \langle \chi | \mathcal{B}(v_M) \cdots \mathcal{B}(v_1) \mathcal{C}(w_1) \cdots \mathcal{C}(w_M) | \chi \rangle \\ &= \frac{\prod_{p=1}^{M} \prod_{q \neq p}^{M} (v_p - w_q)}{\prod_{k>l}^{M}(v_k - v_l)\prod_{i<j}^{M}(w_i - w_j)} \det \mathcal{G} \\ &= \frac{1}{\prod_{k>l}^{M}(v_k - v_l)\prod_{i<j}^{M}(w_i - w_j)} \det \mathcal{F} \end{aligned} \tag{51}$$

where the entries of the $M \times M$ matrices $\mathcal{G}$ and $\mathcal{F}$ are given by

$$\mathcal{G}_{ij} = \left(\mathsf{a}(v_j) - \mathsf{d}(v_j) + \sum_{k \neq i}^{M} \frac{2}{v_j - w_k} \right) \frac{(v_j - w_j)}{(v_j - w_i)^2} \tag{52}$$

$$\mathcal{F}_{ij} = \left(a(v_j) - d(v_j) + \sum_{k\neq i}^{M} \frac{2}{v_j - w_k} \right) \frac{\prod_{l=1}^{M}(v_j - w_l)}{(v_j - w_i)^2}. \tag{53}$$

Above, the parameters $\{w_i\}$ are to satisfy the quasi-classical limit of the Bethe ansatz equations

$$a(w_i) - d(w_i) = \sum_{k\neq i}^{M} \frac{2}{w_k - w_i} \qquad i = 1, \ldots, M \tag{54}$$

while the set $\{v_j\}$ are arbitrary.

Specializing to the case when $\{v_i\} = \{w_i\}$ leads to the formula

$$\mathcal{S}(\vec{v} : \vec{v}) = \det \mathcal{K}$$

where

$$\mathcal{K}_{ii} = a'(v_i) - d'(v_i) - \sum_{k\neq i}^{M} \frac{2}{(v_i - v_k)^2} \qquad \mathcal{K}_{ij} = \frac{2}{(v_i - v_j)^2} \qquad \text{for} \quad i \neq j.$$

We can perform a similar treatment to yield the quasi-classical limit of (49). The terms in η^{2M} give the scalar product of the states in the quasi-classical limit. The terms in η^{2M+1} give not only the form factor for $\mathcal{D}(u)$ but also the next order terms in the expansion of the scalar product, so some care needs to be taken in order to identify the appropriate terms. The result is

$$\{\vec{w}|\mathcal{D}(u)|\vec{v}\rangle = \frac{\Theta(u)}{\prod_{k>l}^{M}(v_k - v_l)\prod_{i<j}^{M}(w_i - w_j)} \left(\left(d(u) - \sum_{p=1}^{M} \frac{1}{u - w_p} \right) \det \mathcal{F} + \sum_{q=1}^{M} \det \mathcal{F}^{(q)} \right) \tag{55}$$

where $\mathcal{F}^{(q)}$ is defined in terms of $\mathcal{F}$ and $\mathcal{Q}$ in analogy with (47) and

$$\mathcal{Q}_{ij}(u) = \frac{\prod_{l\neq j}^{M}(v_j - v_l)}{(u - w_i)^2} \left(a(u) - d(u) + \sum_{k\neq i}^{M} \frac{2}{u - w_k} \right). \tag{56}$$

6. Exact solution of the models

6.1. Solution for the model of two Josephson coupled Bose–Einstein condensates

It is an algebraic exercise to show that the Hamiltonian (1) is related to the matrix $\tilde{t}(u, 0) = \tilde{A}(u, 0) + \tilde{D}(u, 0)$ obtained through (26) via

$$H = -\frac{\mathcal{E}_J}{2}[\tilde{t}(0, 0) - \eta^{-2} + (\alpha N + \beta)^2 - \eta\sigma N - \eta\delta N^2]$$

where we have chosen $\omega(N) = \alpha N + \beta$ and the coupling constants are identified as

$$\begin{aligned} &\eta^2 = \frac{2(U_{11} + U_{22} - U_{12})}{\mathcal{E}_J} \qquad \alpha = \frac{U_{11} - U_{22}}{\eta\mathcal{E}_J} \\ &\beta = \frac{\mu_1 - \mu_2}{\eta\mathcal{E}_J} \qquad \sigma = \frac{\mu_1 + \mu_2}{\eta\mathcal{E}_J} \qquad \delta = \frac{U_{11} + U_{22}}{\eta\mathcal{E}_J}. \end{aligned} \tag{57}$$

Noting that

$$N = \eta^{-1} \frac{\mathrm{d}\tilde{t}}{\mathrm{d}u}(0, 0)$$

the above demonstrates that the Hamiltonian (1) is expressible solely in terms of the matrix $\tilde{t}(u, 0)$ and its derivative.

Since $[H, N] = 0$, the Hamiltonian is block diagonal in the Fock basis (2). Thus on a subspace of the Fock space with fixed particle number N, the diagonalization of $\tilde{t}(u, 0)$ is equivalent to the diagonalization of $t(u, 0)$ presented earlier in the Bethe ansatz framework. It is easily determined that for this case, the total particle number $N = M$ and

$$a(u, N) = u^2 - (\alpha N + \beta)^2 \qquad d(u, N) = \eta^{-2}.$$

From (38) and (39) we deduce the solution of (1) for the energy spectrum to be

$$E(\vec{v}) = -\frac{\mathcal{E}_J}{2}\left[\eta^{-2}\prod_{i=1}^{N}\frac{v_i+\eta}{v_i} - (\alpha N+\beta)^2\prod_{i=1}^{N}\frac{v_i-\eta}{v_i} - \eta^{-2} + (\alpha N+\beta)^2 - \eta\sigma N - \eta\delta N^2\right] \tag{58}$$

where the parameters $\{v_i\}$ are subject to the Bethe ansatz equations

$$\eta^2\left(v_i^2 - (\alpha N+\beta)^2\right) = \prod_{j\neq i}^{N}\frac{v_i - v_j - \eta}{v_i - v_j + \eta}. \tag{59}$$

6.2. *Solution for the model of homo-atomic–molecular Bose–Einstein condensates*

In terms of a realization of the algebra $su(1, 1)$ through

$$K^+ = \frac{(a^\dagger)^2}{2} \qquad K^- = \frac{a^2}{2} \qquad K^z = \frac{2N_a + 1}{4}. \tag{60}$$

one may establish that the relation between the Hamiltonian (3) and the corresponding matrix $\tilde{t}(u, 0) = \tilde{A}(u, 0) + \tilde{D}(u, 0)$ arising from the realization (27) of the Yang–Baxter algebra is

$$H = \sigma + \delta(N/2 + 1/4) + \gamma(N/2 + 1/4)^2 + 2\eta^{-2}\Omega\tilde{t}(0, 0)$$

with

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}u}(0, 0) = 2 - \eta(\eta + \alpha)(N/2 + 1/4) - \eta\beta.$$

Above, we have chosen

$$\begin{aligned}\omega(K^z + N^c) &= \alpha(K^z + N^c) + \beta \\ &= \alpha(N/2 + 1/4) + \beta\end{aligned}$$

and the following identification has been made for the coupling constants:

$$\eta = \frac{4U_{aa} + U_{cc} - 2U_{ac}}{2\Omega} \qquad \alpha = \frac{U_{cc} - 4U_{aa}}{2\Omega} \qquad \beta = \frac{2\mu_c - 4\mu_a + 4U_{aa} - U_{ac}}{4\Omega}$$

$$\sigma = \frac{U_{aa} - 2\mu_a}{4} \qquad \delta = \frac{2\mu_c - U_{ac}}{2} \qquad \gamma = U_{cc}.$$

We deduce

$$a(u, M) = (u + \eta\kappa)\left(1 - \eta u - \eta\left(\alpha(M + \kappa) + \beta\right)\right) \qquad d(u, M) = u - \eta\kappa$$

and by the same argument as in the previous example, we conclude that the exact solution for the energy spectrum of (3) is determined by (38) and (39) which reduces to

$$\begin{aligned}E(\vec{v}) &= \sigma + \delta(M + \kappa) + \gamma(M + \kappa)^2 \\ &\quad + 2\eta^{-1}\kappa\Omega\left[\left(1 - \eta\left(\alpha(M + \kappa) + \beta\right)\right)\prod_{i=1}^{M}\frac{v_i - \eta}{v_i} - \prod_{i=1}^{M}\frac{v_i + \eta}{v_i}\right]\end{aligned} \tag{61}$$

where the parameters $\{v_i\}$ satisfy the Bethe ansatz equations

$$[1-\eta v_i-\eta(\alpha(M+\kappa)+\beta)]\left(\frac{v_i+\eta\kappa}{v_i-\eta\kappa}\right)=\prod_{j\neq i}^{M}\frac{v_i-v_j-\eta}{v_i-v_j+\eta}. \tag{62}$$

For the representation (61) of the $su(1,1)$ algebra there are two lowest weight vectors, namely, the Fock vacuum $|0\rangle$ and the one particle state $a^\dagger|0\rangle$. It follows from (60) that the allowed values for κ in (61) and (62) are $\kappa = 1/4, 3/4$. This demonstrates that the solution of the model depends on whether the total particle number $N = 2M + 2\kappa - 1/2$ is even or odd, the effects of which on the energy spectrum can be seen through numerical analysis (cf [65]).

6.3. Solution for the model of hetero-atomic–molecular Bose–Einstein condensates

In order to show the solvability of the model (4), we adopt the realization of the $su(1,1)$ algebra given by

$$K^+=a^\dagger b^\dagger \qquad K^-=ab \qquad K^z=\frac{N_a+N_b+1}{2} \tag{63}$$

and observe that the operator $\mathcal{I} = N_a - N_b$ commutes with the $su(1,1)$ algebra in this representation, hence taking a constant value in any irreducible representation. Due to the symmetry upon interchanging the labels a and b, we can assume without loss of generality that the eigenvalues of $\mathcal{I}$ are non-negative. In particular, note then that the lowest weight states for this realization are of the form

$$|m\rangle=\frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle \qquad m=0,1,2,\ldots,\infty$$

and $K^z|m\rangle = (m/2+1/2)|m\rangle$. We conclude that the lowest weight labels κ can be taken from the set $\{1/2, 1, 3/2, \ldots\}$ and the eigenvalue of $\mathcal{I}$ in the irreducible representation labelled by κ is $2\kappa - 1$.

For this case the relation between the Hamiltonian (4) and the corresponding matrix $\tilde{t}(u,0)$ from (27) is

$$H=\sigma+\delta(N/2+1/2)+\lambda(N/2+1/2)^2+\rho\mathcal{I}+\nu\mathcal{I}^2+\xi\mathcal{I}(N/2+1/2)+\eta^{-2}\Omega\tilde{t}(0,0) \tag{64}$$

with

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}u}(0,0)=2-\eta(\eta+\alpha)(N/2+1/2)-\eta\gamma\mathcal{I}-\eta\beta.$$

Above, we have chosen

$$\begin{aligned}\omega(K^z+N^c)&=\alpha(K^z+N^c)+\gamma(2\kappa-1)+\beta\\&=\alpha(N/2+1/2)+\gamma\mathcal{I}+\beta\end{aligned}$$

and the coupling constants are related through

$$\begin{aligned}\eta&=\frac{U_{aa}+U_{bb}+U_{cc}+U_{ab}-U_{ac}-U_{bc}}{\Omega}\\ \alpha&=\frac{U_{cc}-U_{aa}-U_{bb}-U_{ab}}{\Omega}\\ \beta&=\frac{2U_{aa}+2U_{bb}+2U_{ab}-U_{ac}-U_{bc}+2\mu_c-2\mu_a-2\mu_b}{2\Omega}\\ \gamma&=\frac{2U_{bb}-2U_{aa}+U_{ac}-U_{bc}}{2\Omega}\\ \sigma&=\frac{U_{aa}+U_{bb}+U_{ab}-2\mu_a-2\mu_b}{4}\end{aligned}$$

$$\delta = \frac{2\mu_c - U_{ac} - U_{bc}}{2}$$
$$\lambda = U_{cc}$$
$$\rho = \frac{U_{bb} - U_{aa} + \mu_a - \mu_b}{2}$$
$$\nu = \frac{U_{aa} + U_{bb} - U_{ab}}{4}$$
$$\xi = \frac{U_{ac} - U_{bc}}{2}.$$

We find

$$a(u, M) = (u + \eta\kappa)\,(1 - \eta u - \eta\,(\alpha(M + \kappa) + \gamma(2\kappa - 1) + \beta)) \qquad d(u, M) = (u - \eta\kappa)$$

and the exact solution in this instance reads

$$E(\vec{v}) = \sigma + \delta(M + \kappa) + \lambda(M + \kappa)^2 + \rho(2\kappa - 1) + \nu(2\kappa - 1)^2 + \xi(2\kappa - 1)(M + \kappa) + \eta^{-1}\kappa\Omega\left[(1 - \eta(\alpha(M + \kappa) + \gamma(2\kappa - 1) + \beta))\prod_{i=1}^{M}\frac{v_i - \eta}{v_i} - \prod_{i=1}^{M}\frac{v_i + \eta}{v_i}\right]$$

where the parameters $\{v_i\}$ satisfy the Bethe ansatz equations

$$[1 - \eta v_i - \eta\,(\alpha(M + \kappa) + \gamma(2\kappa - 1) + \beta)]\left(\frac{v_i + \eta\kappa}{v_i - \eta\kappa}\right) = \prod_{j\neq i}^{M}\frac{v_i - v_j - \eta}{v_i - v_j + \eta}. \tag{65}$$

The total atom number is given by $N = 2M + 2\kappa - 1$.

6.4. Solution for the reduced BCS model

As an alternative to the BCS mean-field approach, one can appeal to the exact solution of the Hamiltonian (5) as given in [38, 39]. Much later it was shown by Cambiaggio *et al* [91] that (5) is integrable in the sense that there exists a set of mutually commutative operators which commute with the Hamiltonian. Our aim here is to show that both these features are consequences of the fact that the Hamiltonian (5) can be derived using the quantum inverse scattering method. This result, which was established in [61, 92], will be proved below. Before doing so, let us remark that there have been several works on this problem, including generalizations [93–95]. In some cases these models can be obtained using trigonometric/hyperbolic versions of the Yang–Baxter algebra. While we will not go into details here, this generalization from the procedure described below is straightforward and is simply a matter of using the trigonometric/hyperbolic analogue of (14) from the outset.

We use a c-number realization $\mathfrak{g}$ of the L-operator, defined by $\mathfrak{g} = \exp(-\alpha\eta\sigma)$ with $\sigma = \mathrm{diag}(1, -1)$, as well as (20) to construct the transfer matrix

$$t(u) = \mathrm{Tr}_0\big(\mathfrak{g}_0 L^S_{0\mathcal{L}}(u - \epsilon_{\mathcal{L}})\cdots L^S_{01}(u - \epsilon_1)\big) \tag{66}$$

which is an element of the $\mathcal{L}$-fold tensor algebra of $su(2)$. Here, Tr_0 denotes the trace taken over the auxiliary space, which for convenience is labelled by 0, while the tensor components of the physical space are labelled $1, \ldots, \mathcal{L}$. Defining

$$T_j = \lim_{u\to\epsilon_j}\frac{u - \epsilon_j}{\eta^2}t(u)$$

for $j = 1, 2, \ldots, \mathcal{L}$, we may write in the quasi-classical limit $T_j = \tau_j + o(\eta)$ and it follows from the commutativity of the transfer matrices that $[\tau_j, \tau_k] = 0, \forall j, k$. Explicitly, these operators read

$$\tau_j = -2\alpha S_j^z + \sum_{k\neq j}^{\mathcal{L}} \frac{\theta_{jk}}{\epsilon_j - \epsilon_k} \tag{67}$$

with $\theta = S^+ \otimes S^- + S^- \otimes S^+ + 2S^z \otimes S^z$. The set of operators (67), realized in terms of canonical fermion operators, are those obtained by Cambiaggio *et al* [91] to establish the integrability of the reduced BCS model. They first appeared in the work of Sklyanin [77] in a general context, and are the Gaudin Hamiltonians [75, 76] in the presence of a non-uniform magnetic field.

Next define a Hamiltonian through

$$H = -\frac{1}{\alpha}\sum_{j=1}^{\mathcal{L}} \epsilon_j \tau_j + \frac{1}{4\alpha^3}\sum_{j,k=1}^{\mathcal{L}} \tau_j \tau_k + \frac{1}{2\alpha^2}\sum_{j=1}^{\mathcal{L}} \tau_j - \frac{1}{2\alpha}\sum_{j=1}^{\mathcal{L}} C_j \tag{68}$$

$$= \sum_{j=1}^{\mathcal{L}} 2\epsilon_j S_j^z - \frac{1}{\alpha}\sum_{j,k=1}^{\mathcal{L}} S_j^+ S_k^- \tag{69}$$

where

$$C = S^+S^- + S^-S^+ + 2(S^z)^2$$

is the Casimir invariant for the $su(2)$ algebra. The Hamiltonian is universally integrable since it is clear that $[H, \tau_j] = 0, \forall j$, irrespective of the realizations of the $su(2)$ algebra in the tensor algebra.

In order to reproduce the Hamiltonian (5) we realize the $su(2)$ generators through the (spin-1/2) hard-core boson representation (6), namely

$$S_j^- = b_j \qquad S_j^+ = b_j^\dagger \qquad S_j^z = \tfrac{1}{2}(n_j - I). \tag{70}$$

In this instance one obtains (5) (up to the constant term $-\sum_{j=1}^{\mathcal{L}} \epsilon_j$) with $g = 1/\alpha$. Incorporating higher spin representations of the $su(2)$ algebra yields models which may be interpreted as coupled BCS systems [93, 94, 96, 97]. Generally, we can define a representation of $su(2)$ through

$$S_j^- = -\sum_{\sigma\in\Omega} a_{j\sigma} a_{j\bar\sigma} \qquad S_j^+ = \sum_{\sigma\in\Omega} a_{j\sigma}^\dagger a_{j\bar\sigma}^\dagger \qquad S_j^z = \frac{1}{2}\sum_{\sigma\in\Omega}(n_{j\sigma} + n_{j\bar\sigma} \pm I) \tag{71}$$

where the operators $a_{j\sigma}, a_{j\sigma}^\dagger$ may be either bosonic (+sign in S_j^z) or fermionic (−sign in S_j^z). Above, $\sigma \in \Omega$ is a degeneracy label and $\bar\sigma \notin \Omega$ refers to the time-reversed state (i.e., the total degeneracy is twice the cardinality of Ω). In this instance one recovers the pairing models discussed in [95, 98, 99]. (For the bosonic case it is convenient to replace $L^S(u)$ with $L^K(u)$ since (71) is not unitary for bosons.) Because of these different possibilities we will derive the eigenvalues of the Hamiltonian (69) in a general setting.

For each index k of the tensor algebra in which the transfer matrix acts, and accordingly in (69), suppose that we represent the $su(2)$ algebra through the irreducible representation with lowest weight (or spin) $-s_k$. Note that we impose no restriction on the allowed values of s_k in order to accommodate infinite-dimensional representations such as the bosonic case of (71). Choosing the pseudovacuum state to be the tensor product of lowest weight states gives

$$a(u) = \exp(-\alpha\eta)\prod_{k=1}^{\mathcal{L}} \frac{u - \epsilon_k - \eta s_k}{u - \epsilon_k} \qquad d(u) = \exp(\alpha\eta)\prod_{k=1}^{\mathcal{L}} \frac{u - \epsilon_k + \eta s_k}{u - \epsilon_k}$$

and the eigenvalues of the transfer matrix (66) as

$$\Lambda(u) = \exp(-\alpha\eta)\prod_{k=1}^{\mathcal{L}}\frac{u-\epsilon_k-\eta s_k}{u-\epsilon_k}\prod_{j=1}^{M}\frac{u-v_j+\eta}{u-v_j} + \exp(\alpha\eta)\prod_{k=1}^{\mathcal{L}}\frac{u-\epsilon_k+\eta s_k}{u-\epsilon_k}\prod_{j=1}^{M}\frac{u-v_j-\eta}{u-v_j}.$$

The corresponding Bethe ansatz equations read

$$\exp(-2\alpha\eta)\prod_{k=1}^{\mathcal{L}}\frac{v_i-\epsilon_k-\eta s_k}{v_i-\epsilon_k+\eta s_k} = \prod_{j\neq i}^{M}\frac{v_i-v_j-\eta}{v_i-v_j+\eta}.$$

The eigenvalues of the conserved operators (67) are obtained through the appropriate terms in the expansion of the transfer matrix eigenvalues in the parameter η. This yields the following result for the eigenvalues λ_j of τ_j:

$$\lambda_j = \left(2\alpha + \sum_{k\neq j}^{\mathcal{L}}\frac{2s_k}{\epsilon_j-\epsilon_k} - \sum_{i=1}^{M}\frac{2}{\epsilon_j-v_i}\right)s_j \tag{72}$$

such that the parameters $\{v_j\}$ satisfy the quasi-classical limit of the Bethe ansatz equations

$$2\alpha + \sum_{k=1}^{\mathcal{L}}\frac{2s_k}{v_i-\epsilon_k} = \sum_{j\neq i}^{M}\frac{2}{v_i-v_j}. \tag{73}$$

For $s_k = 1/2, \forall k$ these equations were found in [38] through a different technique.

Through (72) we can now determine the energy eigenvalues of (69). It is useful to note the following identities:

$$2\alpha\sum_{j=1}^{M}v_j + 2\sum_{j=1}^{M}\sum_{k=1}^{\mathcal{L}}\frac{v_j s_k}{v_j-\epsilon_k} = M(M-1)$$

$$\alpha M + \sum_{j=1}^{M}\sum_{k=1}^{\mathcal{L}}\frac{s_k}{v_j-\epsilon_k} = 0$$

$$\sum_{j=1}^{M}\sum_{k=1}^{\mathcal{L}}\frac{v_j s_k}{v_j-\epsilon_k} - \sum_{j=1}^{M}\sum_{k=1}^{\mathcal{L}}\frac{s_k\epsilon_k}{v_j-\epsilon_k} = M\sum_{k=1}^{\mathcal{L}}s_k.$$

Employing the above it is deduced that

$$\sum_{j=1}^{\mathcal{L}}\lambda_j = 2\alpha\sum_{j=1}^{\mathcal{L}}s_j - 2\alpha M$$

$$\sum_{j=1}^{\mathcal{L}}\epsilon_j\lambda_j = 2\alpha\sum_{j=1}^{\mathcal{L}}\epsilon_j s_j + \sum_{j=1}^{\mathcal{L}}\sum_{k\neq j}^{\mathcal{L}}s_j s_k - 2M\sum_{k=1}^{\mathcal{L}}s_k - 2\alpha\sum_{j=1}^{M}v_j + M(M-1)$$

which, combined with the eigenvalues $2s_j(s_j+1)$ for the Casimir invariants C_j, yields from (68) and (72) the energy eigenvalues

$$E = 2\sum_{j=1}^{M}v_j - 2\sum_{k=1}^{\mathcal{L}}s_k\epsilon_k. \tag{74}$$

From the above expression we see that the quasi-particle excitation energies are given by twice the Bethe ansatz roots $\{v_j\}$ of (73). Finally, let us remark that the eigenstates obtained in taking the quasi-classical limit assume the form

$$|\vec{v}\rangle = \prod_{i=1}^{M} \left(\sum_{j=1}^{\mathcal{L}} \frac{S_j^+}{v_i - \epsilon_j} \right) |0\rangle$$

where $\{v_i\}$ satisfy (73), which are the same as those obtained by Richardson [38] in the case of the reduced BCS model.

An alternative approach to the exact solution of the reduced BCS model, which produces both the eigenstates and the eigenvalues of the conserved operators, was given by Sierra [100] using conformal field theory given by the $SU(2)_k$-WZW model in the limit when the level k approaches -2. There also exists an intriguing analogy for the reduced BCS model from two-dimensional electrostatics [101, 102].

7. Exact calculation of form factors

By using the Slavnov formula (40), (42) and (49), (50) we have explicit determinant representations for the form factors of $A(u), B(u), C(u)$ and $D(u)$. Given any operator, we would like to be able to express it solely in terms of these operators, which we call the inverse problem. Solution of the inverse problem then permits us to determine the form factors for that operator. In the following we will show in several examples how this can be achieved. In some cases we will restrict our analysis to some subclass of the models (1), (3) and (4).

7.1. Form factors for the model of two Josephson coupled Bose–Einstein condensates

Specializing the Slavnov formula (40), (42) to the case of the Hamiltonian (1) gives the matrix elements of F and G as

$$F_{ij} = \frac{\eta^{-1}}{(v_j - w_i)} \left(\left(v_j^2 - (\alpha N + \beta)^2\right) \right) \prod_{k \neq i}^{N} \frac{(v_j - w_k + \eta)}{(v_j - w_k)} - \eta^{-2} \prod_{k \neq i}^{N} \frac{(v_j - w_k - \eta)}{(v_j - w_k)} \Bigg)$$

$$G_{ij} = \frac{\eta^{-1}(v_j - w_j)}{(v_j - w_i)^2} \left(\left(v_j^2 - (\alpha N + \beta)^2\right) \right) \prod_{k \neq i}^{N} \frac{(v_j - w_k + \eta)}{(v_j - w_k)} - \eta^{-2} \prod_{k \neq i}^{N} \frac{(v_j - w_k - \eta)}{(v_j - w_k)} \Bigg).$$

In order to apply the Slavnov formula for the computation of wavefunction norms, we need to determine the functions $\zeta(\vec{v})$, introduced in (44). We can write

$$\begin{aligned} |\vec{v}\rangle &= C(v_1, 1) \cdots C(v_N, N)|0\rangle \\ &= \sum_{k=0}^{N} x_k \left(a_1^\dagger\right)^k \left(a_2^\dagger\right)^{N-k} |0\rangle \end{aligned}$$

for some scalar functions x_i. We deduce from the explicit form of $C(u, j)$ that

$$x_0 = \eta^{-N} \qquad x_N = \prod_{i=1}^{N} (v_i - \alpha N - \beta)\,.$$

On the other hand, we have

$$\begin{aligned} |\vec{v}\} &= B(v_1, 1)^\dagger \cdots B(v_N, N)^\dagger |0\rangle \\ &= \sum_{k=0}^{N} y_k \left(a_1^\dagger\right)^k \left(a_2^\dagger\right)^{N-k} |0\rangle \end{aligned}$$

with

$$y_0 = \prod_{i=1}^{N}(v_i + \alpha N + \beta) \qquad y_N = \eta^{-N}.$$

Using the identity (35) applied to the present case

$$\prod_{i=1}^{N}\left(v_i^2 - (\alpha N + \beta)^2\right) = \eta^{-2N}$$

gives

$$x_0 = \eta^N \prod_{i=1}^{N}(v_i - \alpha N - \beta)\, y_0 \qquad x_N = \eta^N \prod_{i=1}^{N}(v_i - \alpha N - \beta)\, y_N$$

which shows that

$$\zeta(\vec{v}) = \eta^N \prod_{i=1}^{N}(v_i - \alpha N - \beta).$$

The square of the wavefunction norms are then given by

$$\begin{aligned}\|\vec{v}\| &= \langle\vec{v}|\vec{v}\rangle \\ &= \zeta(\vec{v})S(\vec{v} : \vec{v})\end{aligned}$$

where $S(\vec{v} : \vec{v})$ is expressible in terms of F or G.

From (26) we can see that solution to the inverse problem is achieved through

$$a_1^\dagger = \lim_{u\to\infty}\frac{1}{u}C(u) \qquad a_2 = \lim_{u\to\infty}\frac{1}{u}B(u).$$

Using the Slavnov formula we have, for $|\vec{v}\rangle$, $|\vec{w}\rangle$ both eigenstates of the Hamiltonian,

$$\begin{aligned}\langle\vec{v}|a_1|\vec{w}\rangle &= \left\langle\vec{w}\middle|a_1^\dagger\middle|\vec{v}\right\rangle \\ &= \zeta(\vec{w})\left\{\vec{w}\middle|a_1^\dagger\middle|\vec{v}\right\rangle \\ &= \zeta(\vec{w})\lim_{u\to\infty}\frac{1}{u}\{\vec{w}|C(u)|\vec{v}\rangle \\ &= \zeta(\vec{w})\lim_{u\to\infty}\frac{1}{u}S(\vec{w} : v_1, \ldots, v_{M-1}, u) \\ &= \frac{\zeta(\vec{w})\prod_{p=1}^{M-1}\prod_{q\neq p}^{M}(v_p - w_q)}{\prod_{k>l}^{M-1}(v_k - v_l)\prod_{i<j}^{M}(w_i - w_j)}\det\bar{G}\end{aligned}$$

where

$$\begin{aligned}&\bar{G}_{ij} = G_{ij} \qquad \text{for} \quad j \neq M \\ &\bar{G}_{iM} = \eta^{-1}.\end{aligned}$$

In a similar way we find

$$\begin{aligned}\left\langle\vec{w}\middle|a_2^\dagger\middle|\vec{v}\right\rangle &= \langle\vec{v}|a_2|\vec{w}\rangle \\ &= \zeta(\vec{v})\{\vec{v}|a_2|\vec{w}\rangle \\ &= \zeta(\vec{v})\lim_{u\to\infty}\frac{1}{u}\{\vec{v}|B(u)|\vec{w}\rangle \\ &= \zeta(\vec{v})\lim_{u\to\infty}\frac{1}{u}\{\vec{w}|C(u)|\vec{v}\rangle\end{aligned}$$

$$= \zeta(\vec{v}) \lim_{u\to\infty} \frac{1}{u} S(\vec{w} : v_1, \ldots, v_{M-1}, u)$$
$$= \frac{\zeta(\vec{v}) \prod_{p=1}^{M-1} \prod_{q\neq p}^{M} (v_p - w_q)}{\prod_{k>l}^{M-1} (v_k - v_l) \prod_{i<j}^{M} (w_i - w_j)} \det \bar{G}$$

with $\bar{G}$ as above.

In Josephson's original proposal [43, 44] for tunnelling of Cooper pairs through an insulating barrier, the effect is a manifestation of the relative phase difference of the wavefunctions for the two superconductors. Josephson exploited the fact that the BCS variational wavefunction (7) is not an eigenstate of the total particle number, and as phase and particle number are canonically conjugate variables, a well-defined relative phase could be assigned. For the model (1) there are technical difficulties which prevent a simple definition for the phase variable [48, 103]. Consequently, the expectation values for the Josephson tunnelling current

$$\mathcal{J} = \mathrm{i}\big(a_1^\dagger a_2 - a_2^\dagger a_1\big)$$

as well as $\hat{n}$ and $\hat{n}^2$, where $\hat{n} = N_1 - N_2$ is the relative particle number operator, are of primary interest. In principle, these can all be expressed in terms of the form factors for $a_1, a_1^\dagger, a_2$ and $a_2^\dagger$ through completeness relations. This would yield expressions composed of sums of determinants. However, in the case when

$$U_{11} = U_{22} \qquad \mu_1 = \mu_2 \tag{75}$$

which results in $\alpha = \beta = 0$ from (57), we can use a direct method to yield the form factors for $\mathcal{J}$, $\hat{n}$ and $\hat{n}^2$, expressed as single determinants [62]. The reason we can achieve this under the constraint (75) is that in this case the Hamiltonian acquires the additional symmetry

$$[P, H] = 0$$

where P is the permutation operator defined by the action on the Fock basis

$$P \cdot \big(a_1^\dagger\big)^j \big(a_2^\dagger\big)^k |0\rangle = \big(a_1^\dagger\big)^k \big(a_2^\dagger\big)^j |0\rangle.$$

This means that the energy eigenstates are also eigenstates of P, and moreover, $P^2 = I$ shows that P has eigenvalues ± 1. Only by exploiting this symmetry do the form factors for $\hat{n}$, $\hat{n}^2$ and $\mathcal{J}$ become accessible.

As mentioned earlier, the realization of $Y[gl(2)]$ used to derive the model (1) is not unitary. It is however equivalent to a unitary representation when (75) is satisfied in the sense that

$$C^\dagger(u) = PB(u^*)P.$$

Consider for $\{v_i\}$ satisfying the Bethe ansatz equations

$$\begin{aligned}
\langle \vec{v}|\vec{v}\rangle &= \zeta(\vec{v}) S(\vec{v} : \vec{v}) \\
&= \zeta(\vec{v}) \langle 0|B(v_N)\cdots B(v_1)C(v_1)\cdots C(v_N)|0\rangle \\
&= \zeta(\vec{v}) \langle 0|PC^\dagger(v_N^*)P \ldots PC^\dagger(v_1^*)PC(v_1)\ldots C(v_N)|0\rangle \\
&= \zeta(\vec{v}) \langle 0|C^\dagger(v_N^*)\ldots C^\dagger(v_1^*)PC(v_1)\ldots C(v_N)|0\rangle \\
&= \zeta(\vec{v}) \langle \vec{v}|P|\vec{v}\rangle
\end{aligned}$$

which shows that $\zeta(\vec{v}) = \pm 1$ is the eigenvalue of P, namely

$$P|\vec{v}\rangle = \zeta(\vec{v})|\vec{v}\rangle.$$

From the Slavnov formula, the squares of the norms of the eigenstates in this limit

$$\begin{aligned}\|\vec{v}\|^2 &= \langle\vec{v}|\vec{v}\rangle \\ &= |S(\vec{v}:\vec{v})|\end{aligned}$$

are obtained directly.

We define

$$\Xi = A(0) - D(0) = \eta^2 N_1 N_2 + \mathrm{i}\mathcal{J} - \eta^{-2}.$$

Letting $|\vec{v}\rangle$ and $|\vec{w}\rangle$ be the eigenstates of the Hamiltonian we can appeal to (49) and (50) to find

$$\langle\vec{w}|\Xi|\vec{v}\rangle = \frac{-\zeta(\vec{v})\zeta(\vec{w})\eta^{N-2}\prod_{i=1}^{N}(w_i+\eta)}{\prod_{k>l}^{N}(v_k-v_l)\prod_{i<j}^{N}(w_i-w_j)}\det(F+2Q) \tag{76}$$

where the elements of Q read

$$Q_{ij} = \frac{\eta^{-3}\prod_{l=1}^{N}(v_j-v_l-\eta)}{w_i(w_i+\eta)}.$$

We remark that because the basis states are also Hamiltonian eigenstates, it is straightforward to write down the time-dependent form factors

$$\langle\vec{w}|\Xi(t)|\vec{v}\rangle = \exp(-\mathrm{i}t(E(\vec{w})-E(\vec{v})))\langle\vec{w}|\Xi|\vec{v}\rangle \tag{77}$$

where the energies are given by (58), with $\alpha = \beta = 0$.

Remarkably, from equation (76) all the form factors for $\hat{n}, \hat{n}^2$ and $\mathcal{J}$ can be obtained. This is achieved by exploiting the symmetry of the Hamiltonian under P. We begin with the following result, which is easily proved. If $\zeta(\vec{w}) \neq \zeta(\vec{v})$ then

$$\langle\vec{w}|N_1N_2|\vec{v}\rangle = 0.$$

If $\zeta(\vec{w}) = \zeta(\vec{v})$ then

$$\langle\vec{w}|\mathcal{J}|\vec{v}\rangle = 0.$$

The result follows from the observation

$$PN_1N_2 = N_1N_2P \qquad P\mathcal{J} = -\mathcal{J}P.$$

We now find that

$$\langle\vec{w}|N_1N_2|\vec{v}\rangle = \eta^{-2}\langle\vec{w}|\Xi|\vec{v}\rangle + \eta^{-4}\langle\vec{w}|\vec{v}\rangle$$

if $\zeta(\vec{w}) = \zeta(\vec{v})$, and is zero otherwise. Also

$$\langle\vec{w}|\mathcal{J}|\vec{v}\rangle = -\mathrm{i}\langle\vec{w}|\Xi|\vec{v}\rangle$$

if $\zeta(\vec{w}) \neq \zeta(\vec{v})$, and is zero otherwise.

The above shows that the form factors for $\mathcal{J}$ are obtained directly from those of Ξ. Those for $\hat{n}^2$ also follow, since we have $\hat{n}^2 = N^2 - 4N_1N_2$ and the Hamiltonian eigenstates are also eigenstates of the number operator N. Thus

$$\begin{aligned}\langle\vec{w}|\hat{n}^2|\vec{v}\rangle &= N^2\langle\vec{w}|\vec{v}\rangle - 4\langle\vec{w}|N_1N_2|\vec{v}\rangle \\ &= (N^2 - 4\eta^{-4})\langle\vec{w}|\vec{v}\rangle - 4\eta^{-2}\langle\vec{w}|\Xi|\vec{v}\rangle\end{aligned}$$

if $\zeta(\vec{w}) = \zeta(\vec{v})$, and zero otherwise. To obtain the form factors for $\hat{n}$, we use the fact that $\mathcal{J}$ is the time derivative of $\hat{n}$, so

$$\mathcal{J} = \frac{\mathrm{i}}{\mathcal{E}_J}[\hat{n}, H]$$

which gives

$$\langle \vec{w}|\hat{n}|\vec{v}\rangle = \frac{\mathrm{i}\mathcal{E}_J}{E(\vec{w}) - E(\vec{v})}\langle \vec{w}|\mathcal{J}|\vec{v}\rangle$$
$$= \frac{\mathcal{E}_J}{E(\vec{w}) - E(\vec{v})}\langle \vec{w}|\Xi|\vec{v}\rangle$$

if $\zeta(\vec{w}) \neq \zeta(\vec{v})$ and zero otherwise.

The expectation values

$$\langle\theta\rangle_\Psi = \frac{\langle\Psi|\theta|\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

where $\theta = \hat{n}, \hat{n}^2$ or $\mathcal{J}$, and $|\Psi\rangle$ is an arbitrary state, can be expressed in terms of the form factors through completeness relations, in a time-dependent fashion. In particular, for a given $|\Psi\rangle$ the quantum fluctuations of the relative number operator

$$\Delta(\Psi;\hat{n}) = \langle\hat{n}^2\rangle_\Psi - \langle\hat{n}\rangle_\Psi^2$$

can be computed from these results.

The extension of these results to the general case without the imposition of the constraint (75) remains an open problem.

7.2. Form factors for the models of atomic–molecular Bose–Einstein condensates

As both models for atomic–molecular Bose–Einstein condensates are derived from the same L-operator (27), we may treat the two models simultaneously. In analogy with the previous model, we can deduce

$$\zeta(\vec{v}) = \prod_{i=1}^{M}\left(\frac{\eta\kappa + v_i}{\eta\kappa - v_i}\right)$$
$$c^\dagger = \eta^{-1}\lim_{u\to\infty}\frac{1}{u}C(u)$$
$$K^+ = -\eta^{-2}\lim_{u\to\infty}\frac{1}{u}(B(u)^\dagger + C(u)).$$

This leads to the following form factors when $\{w_i\}$ and $\{v_j\}$ both satisfy the Bethe ansatz equations:

$$\begin{aligned}\langle\vec{v},\kappa|c|\vec{w},\kappa\rangle &= \langle\vec{w},\kappa|c^\dagger|\vec{v},\kappa\rangle \\ &= \eta^{-1}\zeta(\vec{w})\lim_{u\to\infty}\frac{1}{u}S(\vec{w}:v_1,\ldots,v_{M-1},u) \\ \langle\vec{v},\kappa|K^-|\vec{w},\kappa\rangle &= \langle\vec{w},\kappa|K^+|\vec{v},\kappa\rangle \\ &= -\eta^{-2}\left(\zeta(\vec{w}) + \zeta(\vec{v})\right)\lim_{u\to\infty}\frac{1}{u}S(\vec{w}:v_1,\ldots,v_{M-1},u).\end{aligned} \tag{78}$$

Note that for this class of models we include the label κ in the Bethe states in order to identify the pseudovacuum used for the Bethe ansatz calculation.

Realizing the $su(1,1)$ algebra in terms of the Heisenberg algebra as in (60) or (63) gives form factors for the models (3) and (4), respectively. Certain form factors for the single particle atomic creation and annihilation operators can also be obtained by using the fact that for these

models there are multiple possible pseudovacuum states for the Bethe ansatz calculations. For the model (3) we have

$$\begin{aligned}\langle\vec{v},1/4|a|\vec{w},3/4\rangle &= \langle\vec{w},3/4|a^\dagger|\vec{v},1/4\rangle\\ &= \langle\vec{w},3/4|\vec{v},3/4\rangle\\ \langle\vec{v},3/4|a|\vec{w},1/4\rangle &= \langle\vec{w},1/4|a^\dagger|\vec{v},3/4\rangle\\ &= \langle\vec{w},1/4|a^\dagger a^\dagger|\vec{v},1/4\rangle\\ &= 2\langle\vec{w},1/4|K^+|\vec{v},1/4\rangle.\end{aligned} \tag{79}$$

In the case of the model (4) we find

$$\begin{aligned}\langle\vec{v},(\kappa-1/2)|a|\vec{w},\kappa\rangle &= \langle\vec{w},\kappa|a^\dagger|\vec{v},(\kappa-1/2)\rangle\\ &= \sqrt{2\kappa-1}\langle\vec{w},\kappa|\vec{v},\kappa\rangle\\ \langle\vec{v},(\kappa+1/2)|b|\vec{w},\kappa\rangle &= \langle\vec{w},\kappa|b^\dagger|\vec{v},(\kappa+1/2)\rangle\\ &= \frac{1}{\sqrt{2\kappa}}\langle\vec{w},\kappa|a^\dagger b^\dagger|\vec{v},\kappa\rangle\\ &= \frac{1}{\sqrt{2\kappa}}\langle\vec{w},\kappa|K^+|\vec{v},\kappa\rangle.\end{aligned} \tag{80}$$

Note that in the case of (79), if $|\vec{v},3/4\rangle$ is an eigenvector of (3), there is no reason to assume that $|\vec{v},1/4\rangle$ is also an eigenvector. Hence formula (78) cannot be used to evaluate (79), since in (78) it is required that $|\vec{v},1/4\rangle$ is an eigenvector. This is because we can only establish that (44) holds for eigenvectors. A similar situation applies to (80).

In the quasi-classical limit the procedure for computing the form acquires a simplified form, in which (79) and (82) can be evaluated. Moreover, the form factors for K^z can be obtained, which are seemingly intractable in the general case. A detailed account is given below. We set the coupling parameters U_{ij} to zero in the Hamiltonians (3) and (4). This corresponds to the ideal gas limit in the sense that the terms with coupling U_{ij} describe the S-wave scatterings between the particles. Mathematically, this means that $\eta = 0$, corresponding to the quasi-classical limit, and $\omega(x) = \beta$ is constant. We scale the generating elements $A(u)$, $B(u)$, $C(u)$, $D(u)$ of (27) by a factor of $1/u$, and in taking the quasi-classical limit we obtain the following realization of the Gaudin algebra:

$$\begin{aligned}\mathcal{A}(u) &= \frac{K^z}{u} - (u+\beta)I \qquad & \mathcal{B}(u) &= \frac{K^-}{u} - c\\ \mathcal{C}(u) &= c^\dagger - \frac{K^+}{u} & \mathcal{D}(u) &= -\frac{K^z}{u}\end{aligned} \tag{81}$$

with

$$\mathsf{a}(u) = \frac{\kappa}{u} - u - \beta \qquad \mathsf{d}(u) = -\frac{\kappa}{u}.$$

This realization is evidently not unitary, but it is clear that

$$\langle\vec{w},\kappa| = (-1)^M\{\vec{w},\kappa|$$

even for arbitrary $\{w_i\}$. By using the quasi-classical limit of the Slavnov formula (51) we may find the scalar product of the states in these models,

$$\begin{aligned}\langle\vec{w},\kappa|\vec{v},\kappa\rangle &= (-1)^M\{\vec{w},\kappa|\vec{v},\kappa\rangle\\ &= \frac{1}{\prod_{k>l}^M(v_k-v_l)\prod_{i<j}^M(w_i-w_j)}\det\bar{\mathcal{F}}\\ &= \frac{\prod_{p=1}^M\prod_{q\neq p}^M(v_p-w_q)}{\prod_{k>l}^M(v_k-v_l)\prod_{i<j}^M(w_i-w_j)}\det\bar{\mathcal{G}}\end{aligned} \tag{82}$$

where we have defined

$$\bar{\mathcal{F}}_{ij} = -\mathcal{F}_{ij} = -\left(a(v_j) - d(v_j) + \sum_{k\neq i}^{M} \frac{2}{v_j - w_k}\right) \frac{\prod_{l=1}^{M}(v_j - w_l)}{(v_j - w_i)^2}$$

$$\bar{\mathcal{G}}_{ij} = -\mathcal{G}_{ij} = -\left(a(v_j) - d(v_j) + \sum_{k\neq i}^{M} \frac{2}{v_j - w_k}\right) \frac{(v_j - w_j)}{(v_j - w_i)^2}$$

in order to absorb the factor $(-1)^M$. The set $\{w_i\}$ provide a solution to the quasi-classical limit of the Bethe ansatz equations

$$w_i + \beta - \frac{2\kappa}{w_i} = \sum_{k\neq i}^{M} \frac{2}{w_i - w_k}. \tag{83}$$

In the quasi-classical limit the energy eigenvalues for the Hamiltonian (3) are

$$E(\vec{v}) = \mu_a(2M + 2\kappa - 1/2) - 2\Omega \sum_{i=1}^{M} w_i$$

while for (4) they are given by

$$E(\vec{v}) = \mu_a(M + 2\kappa - 1) + \mu_b M - \Omega \sum_{i=1}^{M} w_i.$$

In deriving the above energy expressions we have used the identity

$$\beta M + \sum_{i=1}^{M} w_i = \sum_{i=1}^{M} \frac{2\kappa}{w_i}$$

which follows from (83).

For the case when $\{v_j\}$ also satisfy the Bethe ansatz equations we find for the elements of $\bar{\mathcal{F}}$

$$\bar{\mathcal{F}}_{ij} = -\left(\frac{2\kappa}{v_j} - v_j - \beta + \sum_{k\neq i}^{M} \frac{2}{v_j - w_k}\right) \frac{\prod_{l=1}^{M}(v_j - w_l)}{(v_j - w_i)^2}$$

$$= -\left(2\kappa\left(\frac{1}{v_j} - \frac{1}{w_i}\right) + w_i - v_j + \sum_{k\neq i}^{M} \frac{2}{v_j - w_k} - \sum_{k\neq i}^{M} \frac{2}{w_i - w_k}\right) \frac{\prod_{l=1}^{M}(v_j - w_l)}{(v_j - w_i)^2}$$

$$= \left(1 + \frac{2\kappa}{w_i v_j} + \sum_{k\neq i}^{M} \frac{2}{(w_i - w_k)(v_j - w_k)}\right) \prod_{l\neq i}^{M}(v_j - w_l) \tag{84}$$

and similarly

$$\bar{\mathcal{G}}_{ij} = \left(1 + \frac{2\kappa}{w_i v_j} + \sum_{k\neq i}^{M} \frac{2}{(w_i - w_k)(v_j - w_k)}\right) \frac{(v_j - w_j)}{(v_j - w_i)}. \tag{85}$$

Letting $|\vec{v}, \kappa\rangle = |\vec{w}, \kappa\rangle$ gives us the square of the norm formula

$$\|\vec{v}\|^2 = \det \bar{\mathcal{K}}$$

where

$$\bar{\mathcal{K}}_{ii} = 1 + \frac{2\kappa}{v_i^2} + \sum_{k\neq i}^{M} \frac{2}{(v_i - v_k)^2} \qquad \bar{\mathcal{K}}_{ij} = -\frac{2}{(v_i - v_j)^2} \qquad \text{for} \quad i \neq j.$$

To compute the form factors for K^z, we need to take a limit of the form factors of $\mathcal{D}(u)$ as given by (55). This leads us to

$$\begin{aligned}\langle \vec{w}, \kappa | K^z | \vec{v}, \kappa \rangle &= -\lim_{u\to 0} u \langle \vec{w}, \kappa | \mathcal{D}(u) | \vec{v}, \kappa \rangle \\ &= \frac{\kappa \prod_{k=1}^{M} w_k}{\prod_{k>l}^{M}(v_k - v_l) \prod_{i<j}^{M}(w_i - w_j) \prod_{k=1}^{M} v_k} \det(\bar{\mathcal{F}} - \bar{\mathcal{Q}})\end{aligned}$$

with

$$\bar{\mathcal{Q}}_{ij} = -\frac{2\prod_{l\neq j}^{M}(v_j - v_l)}{w_i^2}.$$

Using the fact that $K^z + N_c$ is conserved in both models (3) and (4) and can be expressed in terms of the total atom number N, the form factors for N_c can be deduced from those for K^z.

Next, we turn to the problem of finding the form factors for the operators $c^\dagger$, c, K^+ and K^-. To do this we need to solve the inverse problem and express each of these operators in terms of the realization of the Gaudin algebra. This is not difficult to achieve with the result

$$\begin{aligned} K^+ &= -\lim_{u\to 0} u\mathcal{C}(u) & K^- &= \lim_{u\to 0} u\mathcal{B}(u) \\ c^\dagger &= \lim_{u\to\infty} \mathcal{C}(u) & c &= -\lim_{u\to\infty} \mathcal{B}(u). \end{aligned} \tag{86}$$

Using the fact that the parameters $\{v_i\}$ in the Slavnov formula (82) are arbitrary, we can then take the limits described above to yield the form factors. This gives the results

$$\begin{aligned}\langle \vec{v}, \kappa | K^- | \vec{w}, \kappa \rangle &= \langle \vec{w}, \kappa | K^+ | \vec{v}, \kappa \rangle \\ &= -\lim_{u\to 0} u \langle \vec{w}, \kappa | v_1, \ldots, v_{M-1}, u, \kappa \rangle \\ &= \frac{\prod_{q=1}^{M} w_q}{\prod_{k>l}^{M-1}(v_k - v_l) \prod_{i<j}^{M}(w_i - w_j) \prod_{p=1}^{M-1} v_p} \det \mathcal{P}\end{aligned}$$

where

$$\mathcal{P}_{ij} = \bar{\mathcal{F}}_{ij} \qquad \mathcal{P}_{iM} = -\frac{2\kappa}{w_i^2} \qquad \text{for} \quad j \neq M$$

and

$$\begin{aligned}\langle \vec{v}, \kappa | c | \vec{w}, \kappa \rangle &= \langle \vec{w}, \kappa | c^\dagger | \vec{v}, \kappa \rangle \\ &= \lim_{u\to\infty} \langle \vec{w}, \kappa | v_1, \ldots, v_{M-1}, u, \kappa \rangle \\ &= \frac{\prod_{p=1}^{M-1} \prod_{q\neq p}^{M}(v_p - w_q)}{\prod_{k>l}^{M-1}(v_k - v_l) \prod_{i<j}^{M}(w_i - w_j)} \det \mathcal{W}\end{aligned}$$

where

$$\begin{aligned} &\mathcal{W}_{ij} = \bar{\mathcal{G}}_{ij} \qquad \text{for} \quad j \neq M \\ &\mathcal{W}_{iM} = 1. \end{aligned}$$

7.3. Form factors for the reduced BCS model

The results of this section have been published in [61] for the case $s_k = 1/2, \forall k$ (although different conventions and notation were used). A closely related study is given in [104]. The fundamental difference between [104] and the results given below is that [104] employs the generating function of correlators of the Gaudin algebra as developed in [31], whereas below we will directly use the quasi-classical limit of the Slavnov formula as given by (51). By this procedure the form factors are obtained in an explicit determinant representation. Again, we will derive results for the general case of the Hamiltonian (69) where the irreducible realizations of the $su(2)$ algebras, labelled by a lowest weight $-s_k$, are arbitrary. The realization of the Gaudin algebra obtained by taking the quasi-classical limit of the realization of the Yang–Baxter algebra given by (68) reads

$$\begin{aligned} \mathcal{A}(u) &= -\alpha I + \sum_{k=1}^{\mathcal{L}} \frac{S_k^z}{u-\epsilon_k} \qquad & \mathcal{B}(u) &= \sum_{k=1}^{\mathcal{L}} \frac{S_k^-}{u-\epsilon_k} \\ \mathcal{C}(u) &= \sum_{k=1}^{\mathcal{L}} \frac{S_k^+}{u-\epsilon_k} & \mathcal{D}(u) &= \alpha I - \sum_{k=1}^{\mathcal{L}} \frac{S_k^z}{u-\epsilon_k} \end{aligned} \tag{87}$$

with

$$a(u) = -\alpha - \sum_{k=1}^{\mathcal{L}} \frac{s_k}{u-\epsilon_k} \qquad d(u) = \alpha + \sum_{k=1}^{\mathcal{L}} \frac{s_k}{u-\epsilon_k}.$$

This realization is unitary, so we do not need to deal with the issues of non-unitarity as in the previous examples.

By using the quasi-classical limit of the Slavnov formula (51) we find the scalar product of the states

$$\begin{aligned} \langle \vec{w} | \vec{v} \rangle &= \frac{1}{\prod_{k>l}^{M}(v_k - v_l) \prod_{i<j}^{M}(w_i - w_j)} \det \mathcal{F} \\ &= \frac{\prod_{p=1}^{M} \prod_{q\neq p}^{M}(v_p - w_q)}{\prod_{k>l}^{M}(v_k - v_l) \prod_{i<j}^{M}(w_i - w_j)} \det \mathcal{G} \end{aligned} \tag{88}$$

where $\mathcal{G}_{ij}$ and $\mathcal{F}_{ij}$ are given by (52) and (53) and the Bethe ansatz equations for the parameters $\{w_i\}$ are given by (73). Letting $\{v_i\}$ also be a solution of the Bethe ansatz equations we find

$$\begin{aligned} \mathcal{F}_{ij} &= \left(-2\alpha - \sum_{k=1}^{\mathcal{L}} \frac{2s_k}{v_j - \epsilon_k} + \sum_{k\neq i}^{M} \frac{2}{v_j - w_k} \right) \frac{\prod_{l=1}^{M}(v_j - w_l)}{(v_j - w_i)^2} \\ &= \left(\sum_{k=1}^{\mathcal{L}} \frac{2s_k}{(w_i - \epsilon_k)(v_j - \epsilon_k)} - \sum_{k\neq i}^{M} \frac{2}{(w_i - w_k)(v_j - w_k)} \right) \prod_{l\neq i}^{M}(v_j - w_l) \\ \mathcal{G}_{ij} &= \left(\sum_{k=1}^{\mathcal{L}} \frac{2s_k}{(w_i - \epsilon_k)(v_j - \epsilon_k)} - \sum_{k\neq i}^{M} \frac{2}{(w_i - w_k)(v_j - w_k)} \right) \frac{(v_j - w_j)}{(v_j - w_i)}. \end{aligned}$$

Setting $|\vec{v}\rangle = |\vec{w}\rangle$ gives us the square of the norm formula

$$\|\vec{v}\|^2 = \det \mathcal{K}$$

where

$$\mathcal{K}_{ii} = \sum_{k=1}^{\mathcal{L}} \frac{2s_k}{(v_i - \epsilon_k)^2} - \sum_{k\neq i}^{M} \frac{2}{(v_i - v_k)^2} \qquad \mathcal{K}_{ij} = \frac{2}{(v_i - v_j)^2} \qquad \text{for} \quad i \neq j.$$

Putting $s_k = 1/2, \forall k$, the above is exactly the norm square formula obtained by Richardson [105].

To compute the form factors for S_m^z, we use the fact that $\mathcal{D}(u)$ has simple poles at $u = \epsilon_j, \forall j$, i.e.,

$$S_m^z = -\lim_{u\to\epsilon_m}(u-\epsilon_m)\mathcal{D}(u).$$

This leads to

$$\begin{aligned}\left\langle \vec{w}\middle| S_m^z |\vec{v}\right\rangle &= -\lim_{u\to\epsilon_m}(u-\epsilon_m)\langle\vec{w}|\mathcal{D}(u)|\vec{v}\rangle \\ &= \frac{-s_m\prod_{k=1}^{M}(w_k-\epsilon_m)}{\prod_{k=1}^{M}(v_k-\epsilon_m)\prod_{k>l}^{M}(v_k-v_l)\prod_{i<j}^{M}(w_i-w_j)}\det(\mathcal{F}-\mathcal{Q}(\epsilon_m))\end{aligned}$$

where $\mathcal{F}$ is as given above and

$$\mathcal{Q}_{ij}(u) = \frac{2\prod_{l\neq j}^{M}(v_j-v_l)}{(u-w_i)^2}.$$

Now we derive the form factors for the operators S^+ and S^-. In this instance the inverse problem is solved as follows:

$$S_m^- = \lim_{u\to\epsilon_m}(u-\epsilon_m)\mathcal{B}(u) \qquad S_m^+ = \lim_{u\to\epsilon_m}(u-\epsilon_m)\mathcal{C}(u). \tag{89}$$

Using the fact that the parameters $\{v_i\}$ in the Slavnov formula (88) are arbitrary, we can then take the limits described above to yield the form factors. The results are

$$\begin{aligned}\langle\vec{v}|S_m^-|\vec{w}\rangle &= \langle\vec{w}|S_m^+|\vec{v}\rangle \\ &= \lim_{u\to\epsilon_m}(u-\epsilon_m)\langle\vec{w}|v_1,\ldots,v_{M-1},u\rangle \\ &= \frac{\prod_{q=1}^{M}(w_q-\epsilon_m)}{\prod_{p=1}^{M-1}(v_p-\epsilon_m)\prod_{k>l}^{M-1}(v_k-v_l)\prod_{i<j}^{M}(w_i-w_j)}\det\mathcal{P}\end{aligned}$$

where

$$\begin{aligned}\mathcal{P}_{ij} &= \mathcal{F}_{ij} \qquad \text{for} \quad j\neq M \\ \mathcal{P}_{iM} &= \frac{1}{(w_i-\epsilon_m)^2}.\end{aligned}$$

The above form factors can be used to construct general correlation functions, such as the Penrose–Onsager–Yang off-diagonal long-range order parameter as given in [61].

8. Conclusion

We have reviewed the theory of the quantum inverse scattering method and algebraic Bethe ansatz for the computation of energy spectra and form factors in exactly solvable models, and demonstrated how it applies to several models of Bose–Einstein condensates and the reduced BCS model. Throughout we have only used the specific example of the Yang–Baxter algebra associated with the Lie algebra $gl(2)$. However, a Yang–Baxter algebra can be associated with any simple Lie algebra, Lie superalgebra, and the q-deformations of these structures. Hence the theory can be applied on a much wider level. For example, generalized BCS systems derived from Yang–Baxter algebras associated with the Lie algebras $gl(4)$ and $so(5)$ and the Lie superalgebra $gl(2|1)$ were derived in [106–108], respectively. The model obtained in [107] was proposed by Richardson in 1966 to describe proton–neutron pairing in nuclear

systems [109, 110]. The case of a general Lie algebra was examined in [111] in the context of the Knizhnik–Zamolodchikov equation. One major challenge which remains is to extend the Slavnov formula for the scalar products of states to the general case. In fact there has been little progress on this aspect with the exceptions of the work by Reshetikhin on the norms of the wavefunctions for models derived from the $gl(3)$ Yang–Baxter algebra [112] and by Göhmann and Korepin on the Hubbard model [113]. Another approach based on the Knizhnik–Zamolodchikov equation for $gl(n)$ can be found in [114].

In the method we have described the exact solution is parametrized in terms of the Bethe ansatz equations, which cannot be solved analytically. Consequently, numerical analysis of the solutions must be undertaken. For the BCS model there have been quite a number of works on this topic (e.g., see [40, 102, 104, 115, 116]). For the models of Bose–Einstein condensates the only numerical analysis of an exact solution, of which we know, is in [65]. It is also possible to conduct an asymptotic analysis of the Bethe ansatz equations to find the exact asymptotic behaviour of the energy spectrum and correlation functions. Examples are given in [63–65].

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